Stability Analysis of Uncertain Discrete-Time Piecewise Linear Systems with Time Delays

Ou Ou, Hong-Bin Zhang, and Ju-Bang Yu

Abstract—This paper considers the stability analysis of uncertain discrete-time piecewise linear systems with time delays based on piecewise Lyapunov–Krasovskii functionals. It is shown that the stability can be established for the control systems if there is a piecewise Lyapunov–Krasovskii functional, and moreover, the functional can be obtained by solving a set of linear matrix inequalities (LMIs) that are numerically feasible. A numerical example is given to demonstrate the efficiency and advantage of the proposed method.

Index Terms—Discrete-time, linear matrix inequalities, piecewise linear systems, stability, time-delay systems.

1. Introduction

Piecewise-linear systems have been a subject of research in control system community[1][9]. On one hand, piecewise-linear systems constitute a special class of hybrid systems and often arise in control systems when piecewise-linear components are encountered. These components include dead-zone, saturation, relays, and hysteresis. On the other hand, many other classes of nonlinear systems can be approximated by the piecewise-linear systems, thus the piecewise-linear systems can provide a powerful means of analysis and design for more general nonlinear control systems.

A number of results have been obtained recently on analysis and design of such piecewise-linear systems. For example, Imura et al. studied the well-posedness of piecewise-linear systems[1]. A number of necessary or sufficient conditions for well-posedness have been proposed. Johansson and Rantzer et al. presented results on stability and optimal performance analysis for piecewise-linear systems based on a piecewise-continuous Lyapunov function[2][13]. It was shown that lower bounds and upper bounds of the optimal control cost can be obtained by semidefinite programming, and the framework of piecewise-linear systems can be used to analyze smooth nonlinear systems with arbitrary accuracy. References [9] to [15] presented results on robust stability analysis, $H_\infty$ control for continuous-time piecewise-linear systems, results on stability analysis, $H_\infty$ control and $H_\infty$ filtering for discrete-time piecewise linear systems.

It is well known that there are time delays in practical complex control systems, so time-delay piecewise linear control systems are studied recently. Kulkarni investigated the stability of continuous-time piecewise linear systems with time delays based on Piecewise Quadratic Lyapunov Functionals[13]. The authors in [16] considered the $H_\infty$ filtering of discrete-time piecewise systems with time delays based on the piecewise Lyapunov-Krasovskii functionals. To the best of our knowledge, the problem of stability analysis has not been addressed for uncertain discrete-time piecewise linear systems with both parametric uncertainties and time delays, which is very challenging and remains open.

In this paper, we present a new stability analysis method for uncertain discrete-time piecewise-linear systems with time delays based on piecewise Lyapunov-Krasovskii functionals. Via the proposed piecewise Lyapunov-Krasovskii functional based approach, the conservatism arising from common Lyapunov-Krasovskii functional based stability analysis of discrete-time piecewise linear systems with time delays can be relaxed. Moreover, the stability checking result and the control laws can be obtained by solving a set of LMIs that are numerically tractable with commercially available software.

The rest of this paper is organized as follows. System descriptions and preliminaries are presented in Section 2. The stability analysis of uncertain discrete-time piecewise linear systems with time delays is formulated in Section 3. In Section 4, a simulation example is provided to demonstrate effectiveness of our method. Finally, conclusions are given in Section 5.

Notations: the notations used are fairly standard. The square norm of a vector $z \in \mathbb{R}^n$ is denoted by $z^T z$ or $\|z\|_2^2$. The notation $l_z[0,T]$ is used for vector-valued functions, that is, we say that $z:[0,N] \to \mathbb{R}^n$ is in $l_z[0,N]$ if $\sum_{k=0}^{N} |z(k)|^2 < \infty$ and its $l_z$-norm is defined as $\|z\|_z = \left(\sum_{k=0}^{N} |z(k)|^2 \right)^{1/2}$. The notation $X \succeq 0 \ (X > 0)$
means that the matrix $X$ is positive semidefinite (positive definite). $I$ is the identity matrix with appropriate dimensions. The symbol "\text{sym}" in a matrix $A \in \mathbb{R}^{n \times n}$ stands for the transposed elements in the symmetric positions. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. System Descriptions and Preliminaries

Consider an uncertain discrete-time piecewise linear system with time delay described by the following equation:

$$
x(k+1) = [A_{x} + \Delta A_{y}(k)]x(k) + [A_{y} + \Delta A_{x}(k)]x(k-d), \quad x(k) \in X_{l} \quad (1)
$$

where $X_{l} \in \mathbb{R}^{n}$ signifies a partition of the state space into a number of closed polyhedral subspaces, $L$ is the index set of subspaces, $x(k) \in \mathbb{R}^{n}$ is the state vector of the system, $x(k-d) \in \mathbb{R}^{n}$ is the delay state vector with constant delay $d > 0$, $(A_{x}, A_{y})$ is the $l$th local model of the system, and $(\Delta A_{x}(k), \Delta A_{y}(k))$ are the uncertainty terms of the $l$th local model of the system. In this paper, the uncertainty terms are assumed to be of the form:

$$
[\Delta A_{x}(k) \ \Delta A_{y}(k)] = M_{l}F(k)[N_{l1} \ \ N_{l2}] \quad (2)
$$

where $M_{l}, N_{l1}, N_{l2}, N_{d1}$, and $N_{d2}$ are known as real constant matrices and $F(k)$ is an unknown time-varying matrix function satisfying

$$
F(k)^{T}F(k) \leq I \quad \forall k . \quad (3)
$$

It is assumed that all the elements of $F(k)$ are Lebesgue measurable.

For convenient notation, we introduce

$$
\bar{A}_{x} = A_{x} + \Delta A_{x}(k), \quad \bar{A}_{y} = A_{y} + \Delta A_{y}(k) .
$$

Then, using this notation, the system model (1) can be expressed as

$$
x(k+1) = \bar{A}_{x}x(k) + \bar{A}_{y}x(k-d), \quad x(k) \in X_{l} . \quad (4)
$$

To end this section, the following lemma is stated, which is useful for proving the main results in the subsequent sections.

Lemma 1: Given matrices $P$, $M$, and $N$ of appropriate dimensions and with $P$ symmetrical, then

$$
P + MF(k)N + N^{T}F^{T}(k)M^{T} < 0
$$

holds for all of the $F(k)$ satisfying $F(k)^{T}F(k) \leq I$, if and only if for some $\epsilon > 0$, $P + \epsilon^{-1}MM^{T} + \epsilon N^{T}N < 0$.

3. Robust Stability

In this section, the stability analysis of the systems described in the last section is discussed. The stability condition for the system without control input and external disturbance can be summarized in the following theorem.

Theorem 1 (Stability Analysis). The system (1), or equivalently (4) with $u = v = 0$, is asymptotically stable if there exist matrices $P_{l} > 0$, matrices $Q > 0$, $R$, and positive constant $\epsilon_{l}$, satisfying

$$
\begin{bmatrix}
P_{l} - R - R^{T} & RA_{x} & RA_{y} & RM_{l} & 0 \\
* & -P_{l} + Q & 0 & 0 & \epsilon_{l}N_{l1}^{T} \\
* & * & -Q & 0 & \epsilon_{l}N_{l2}^{T} \\
* & * & * & -\epsilon_{l}I & 0 \\
* & * & * & * & -\epsilon_{l}I
\end{bmatrix} < 0 \quad (5)
$$

for $l \in L$, $l' \in L$.

Proof. Consider the following piecewise Lyapunov-Krasovskii functional $V(k)$ for the discrete-time piecewise linear system (4):

$$
V(k) = \bar{x}^{T}(k)P_{l}x(k) + \sum_{i=1}^{k} \bar{x}^{T}(i)Q_{i}x(i), \quad x(k) \in X_{l}, \quad l \in L . \quad (6)
$$

Without loss of generality, assume that $x(k) \in X_{l}$ and $x(k+1) \in X_{l'}$, then

$$
\Delta V(k) = \bar{x}^{T}(k)P_{l}x(k+1) - \bar{x}^{T}(k)P_{l}x(k)
$$

$$
= \bar{x}^{T}(k)(P_{l} - \bar{A}_{x} - \bar{A}_{y} - Q)\bar{x}(k)
$$

$$
= \bar{x}^{T}(k)(\bar{A}_{x}P_{l}\bar{A}_{x} - P_{l} + M)\bar{x}(k) \quad (7)
$$

where

$$
\bar{x}(k) = [\bar{x}(k) \ \ x^{T}(k-d)]^{T}, \quad M = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}
$$

$$
\bar{A}_{l} = \begin{bmatrix} \bar{A}_{x} & \bar{A}_{y} \\ \bar{A}_{d} & \bar{A}_{d} \end{bmatrix}, \quad \bar{P}_{l} = \begin{bmatrix} P_{l} & 0 \\ 0 & 0 \end{bmatrix} .
$$

Apply the Schur complement [17] to (5) yields

$$
\begin{bmatrix}
P_{l} - R - R^{T} & RA_{x} & RA_{y} & RM_{l} & 0 \\
* & -P_{l} + Q & 0 & 0 & \epsilon_{l}N_{l1}^{T} \\
* & * & -Q & 0 & \epsilon_{l}N_{l2}^{T} \\
* & * & * & -\epsilon_{l}I & 0 \\
* & * & * & * & -\epsilon_{l}I
\end{bmatrix} + \epsilon_{l}N_{l1}^{T}N_{l2}^{T} < 0 . \quad (8)
$$

It follows from Lemma 1 that
\[
\begin{bmatrix}
P_f - R - R^T & R A_1 & R A_d \\
\ast & -P_f + Q & 0 \\
\ast & \ast & -Q
\end{bmatrix} < 0. \tag{9}
\]

Premultiplying and postmultiplying (9) by \(P_f\) and its transposed matrix respectively, we have
\[
\begin{bmatrix}
P_f & A_1 + Q & A_d \\
A_f^T P_f & -P_f + A_d^T A_d & -Q \\
A_d^T P_f & A_d^T A_d - Q
\end{bmatrix} < 0 \tag{10}
\]

which means
\[
\dot{A}^T P_f A_d - A_d^T P_f A_f < 0.
\]

According to (7), \(\Delta V(k) < 0\) for all \(\Delta t(k) \neq 0\). We conclude that the open loop system is asymptotically stable.

**Remark 1.** It should be noted that when \(P_f = P_d = P\), Theorem 1 is reduced to the common Lyapunov-Krasovskii functional based stability condition. In fact, the common Lyapunov-Krasovskii functional is a special case of the more general piecewise Lyapunov-Krasovskii functional.

4. **Numerical Example**

Consider the following uncertain discrete-time piecewise linear system with time delay
\[
x(k+1) = [A_j + \Delta A_j(k)]x(k) + [A_d + \Delta A_d(k)]x(k-d) \\
x(k) \in X, l = 1, 2, 3. \tag{11}
\]

where \(x(k) = [x_1(k), x_2(k)]^T\) is the state vector. \(x(k-d) \in \mathbb{R}^n\) is the state vector with constant delay \(d > 0\). The cell partitions for the system are \(X_i, X_2, X_3\), respectively. The parameters of the system matrices are given as follows

\[
\begin{align*}
A_1 &= \begin{bmatrix}
-0.2 & 0.5 \\
-0.3 & -1.1
\end{bmatrix}, & A_2 &= \begin{bmatrix}
-0.4 & 0.5 \\
-0.2 & 0.1
\end{bmatrix} \\
A_3 &= \begin{bmatrix}
-0.55 & 0.5 \\
-0.3 & 0.1
\end{bmatrix}, & A_d &= \begin{bmatrix}
-0.1 & 0.05 \\
0 & 0.1
\end{bmatrix} \\
A_{d2} &= \begin{bmatrix}
-0.1 & 0.05 \\
0 & -0.1
\end{bmatrix}, & A_d &= \begin{bmatrix}
-0.1 & 0.05 \\
0 & 0.05
\end{bmatrix} \\
M &= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, & N_{ij} &= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}. \tag{11}
\end{align*}
\]

When \(\alpha = 0.2\), using Theorem 1, a piecewise Lyapunov-Krasovskii functional \(V(k)\) can be found so that the asymptotic stability of the origin of the open loop system is ensured for all constant delay \(d > 0\). And the solutions to those LMIs are given below

\[
P_1 = \begin{bmatrix}
64.3215 & -17.3496 \\
-17.3496 & 95.2818
\end{bmatrix}, \quad P_2 = \begin{bmatrix}
69.6379 & -26.5831 \\
-26.5831 & 90.8286
\end{bmatrix}
\]

\[
P_3 = \begin{bmatrix}
76.6808 & -32.7685 \\
-32.7685 & 84.6450
\end{bmatrix}, \quad Q = \begin{bmatrix}
24.4531 & -5.0148 \\
-5.0148 & 27.6631
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
69.9327 & -23.3598 \\
-23.3598 & 82.2055
\end{bmatrix}, \quad \varepsilon_1 = 9.9836
\]

\[
\varepsilon_2 = 9.2317, \quad \varepsilon_3 = 8.2944.
\]

If we consider the common Lyapunov-Krasovskii functional for stability analysis; i.e., \(P_f = P_d = P\) and \(\varepsilon = \varepsilon_1\) for all \(l \in L\) in LMIs (5), there is no feasible solution to those LMIs. So, the common Lyapunov-Krasovskii functional method can not be used to show the stability of this system. Thus it can be expected that the proposed stability analysis approach based on the piecewise Lyapunov-Krasovskii functional is less conservative but more powerful than the existing approaches based on the common Lyapunov-Krasovskii functional.

5. **Conclusions**

In this paper, a new method is presented to design \(H_\infty\) controller for uncertain discrete-time piecewise linear systems with time delays based on piecewise Lyapunov-Krasovskii functionals. It is shown that the stability and control synthesis results based on the piecewise Lyapunov-Krasovskii functionals are less conservative but more powerful than those based on the common Lyapunov-Krasovskii functionals. A numerical example is presented to demonstrate the advantages of the proposed approach.

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**References**


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