Complicated Attractor and Novel Route to Chaos via Cubic Nonlinearity Controller

Xiao-Hua Qiao

Abstract—By introducing a nonlinear state feedback controller, a complicated chaotic attractor of four-dimensional continuous autonomous chaotic system evolved from Lorenz system is proposed. The controller consists of a cubic cross-product term of three variables and a self variable term with control parameter. This paper verifies that the proposed system has complex dynamical behavior, which can evolve into periodic and chaotic behavior cyclically. These circular routes to chaos have self-similar bifurcation structure in a wide parameter variation range. An electronic circuit is designed to demonstrate the complicated attractor.

Index Terms—Attractor, chaotic system, cubic nonlinearity controller, bifurcation route.

1. Introduction

Recently, chaos has been found to be very useful and has great potential in many technological disciplines such as in information and computer sciences, power systems protection, biomedical systems analysis, flow dynamics and liquid mixing, encryption and communications, and so on\(^{[1]-[3]}\). There has been increasing interest in exploiting chaotic dynamics in engineering applications, where some attention has been focused on effectively creating chaos via simple physical systems such as electronic circuits and switching piecewise-linear controllers\(^{[4]-[6]}\).

Lorenz system is a famous chaotic system of three-dimensional autonomous ordinary differential equations\(^{[7]}\), which has two quadratic nonlinearity terms and is described by

\[
\begin{align*}
\frac{dx}{dt} &= a(y - x) \\
\frac{dy}{dt} &= cx - y - xz \\
\frac{dz}{dt} &= xy - bz
\end{align*}
\]

when \(a=10\), \(b=8/3\), \(c=28\), system (1) has a chaotic attractor. Since this system was proposed, much attention had been paid to find out new chaotic systems, such as Rössler system\(^{[8]}\), Chen system\(^{[9]}\), Lü system\(^{[10]}\), and other new chaotic systems\(^{[11],[15]}\), among which Rössler’s attractor has a single folded-band structure, its one-scroll structure is the simplest topological structure for a three-dimensional quadratic autonomous chaotic system, while Lorenz, Chen’s and Lü’s attractor with two-scroll structures are topologically more complex than Rössler’s attractor.

In this paper, we will introduce a new four-dimensional autonomous dissipative system of differential equations by adding a nonlinear state feedback controller with cross-product term of three variables to Lorenz system. This new system can display a 2-scroll attractor with complex topological structure. There is little report about its algebraic system structure with a cubic nonlinear term\(^{[15]}\). The chaotic characteristics of the cubic system are verified by calculating its Lyapunov exponent spectrum and investigating its bifurcation route. Various complicated attractors and a novel bifurcation route evolving from period to chaos, then jumping into period circularly, can be observed. This new system with complex dynamical behavior can be applied to improve the security of chaotic secrecy systems and chaotic information encryption.

2. New Chaotic System via Cubic Controller

2.1 Complicated Chaotic Attractor

Based on system (1), one can construct a new fourth-order chaotic system by introducing a nonlinear state feedback controller as given below

\[
\begin{align*}
\frac{dx}{dt} &= a(y - x) \\
\frac{dy}{dt} &= cx - y - 10xz \\
\frac{dz}{dt} &= 10xz - bz + 10w \\
\frac{dw}{dt} &= 10xz - dw
\end{align*}
\]

where \(a\), \(b\), \(c\), and \(d\) are real constants.

This system is found to be chaotic in a very wide parameters range and has many interesting complex dynamical behaviors. When \(a=10\), \(b=8/3\), \(c=28\), and \(d=8\),
system (2) is chaotic and displays a complicated 2-scroll chaotic attractor, as shown in Fig. 1 (a) to (c). The Lyapunov exponents of system (2) are \( LE_1 = 0.6489, LE_2 = 0, LE_3 = -8.6732, LE_4 = -13.6289 \), and the Lyapunov dimension is \( d_L = 2.029 \) for initial value \((1, 1, 0, 0)\). Fig. 2 shows the Poincaré mappings on \( z = 6 \) section for the corresponding parameters, with several sheets of the attractors visualized. It is clear that these sheets are strange with special folded.

The system (2) is symmetrical about \( z-w \) plane and dissipative. The symmetrical property can be found from the invariance of system (2) under the transformation \((x, y, z, w) \rightarrow (-x, -y, z, w)\). The dissipativity is derived from

\[
\nabla V = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} + \frac{\partial V}{\partial w} = -(a+b+d+1). 
\]

Therefore, all system orbits are ultimately confined to a specific subset of zero volume, and the asymptotic motion settles onto an attractor.

### 2.2 Equilibrium and Stability

The Jacobian matrix of system (2) at the equilibrium \( S^* = (x^*, y^*, z^*, w^*) \) is given by

\[
J_{S^*} = \begin{bmatrix}
-a & a & 0 & 0 \\
-c-10z^* & -1 & -10x^* & 0 \\
10y^* & 10x^* & -b & 10 \\
10y^*z^* & 10x^*z^* & 10x^*y^* & -d \\
\end{bmatrix}
\]

Clearly, if \( b, d > 0 \) and \( c > 1 \), the system has three equilibria, as below:

\[
S_0 = (0, 0, 0, 0) \\
S_+ = \left[ \sqrt{\delta d}, \sqrt{\delta d}, c-1, \delta(c-1) \right] \\
S_- = \left[ -\sqrt{\delta d}, -\sqrt{\delta d}, c-1, \delta(c-1) \right]
\]

where \( \delta = 0.01b(c-1) \).

For the origin \( S_0 \), system (2) has four characteristic roots:

\[
\lambda_1 = -a + \frac{1}{2} \sqrt{(a+1)^2 + 4a(c-1)} \\
\lambda_2 = -a + \frac{1}{2} \sqrt{(a+1)^2 + 4a(c-1)} \\
\lambda_3 = -b \\
\lambda_4 = -d - 1.
\]

It is clear that \( \lambda_1 > 0, \lambda_2, 3, 4 < 0 \) for \( a, b, d > 0 \) and \( c > 1 \), therefore, the origin is a saddle point in three-dimension.

The other equilibria \( S_+ \) and \( S_- \) are symmetrically placed with respect to \( z-w \) plane. For these equilibria, when \( a = 10, b = 8/3, c = 28 \), system (3) yields following characteristic equation

\[
\lambda^4 + \left( \frac{41}{3} + d \right) \lambda^3 + \left( \frac{88}{3} + \frac{41}{3} d \right) \lambda^2 + \frac{88}{3} + \frac{72(36 + d)}{27 + d} \left( d \lambda + \frac{72d(540 + 20d)}{27 + d} \right) = 0
\]

The coefficients of this polynomial equation are all positive, then by Routh-Hurwitz criterion, for \( d > 0 \), two non-zero equilibria \( S_+ \) and \( S_- \) are both unstable. When \( d = 8 \), the characteristic roots of system (2) for \( S_+ \) and \( S_- \) are \( \lambda_{1,2} = 2.0133 \pm 7.8109i \) and \( \lambda_{3,4} = -12.8466 \pm 3.4672i \), therefore, the equilibria \( S_+ \) and \( S_- \) are all saddle-foci, Hopf bifurcation can appear at these equilibria.

### 3. Novel Route to Chaos

The Lyapunov exponent spectrum and the corresponding
bifurcation diagram of state variable $y$ versus parameter $d$ are shown in Fig. 3, where $a = 10$, $b = 8/3$ and $c = 28$. For clarity, the first two Lyapunov exponents are presented.

From Fig. 3, it can be seen that chaotic attractor and limit cycle appear alternatively in the whole parameter region, and the band widths of chaos and period become wider and wider with the increasing of $d$. As $d$ increases, system (2) exhibits period-doubling bifurcation route, and evolves from the orbit of period, such as period-2, period-4, period-8, etc., to the orbit of chaos, and then suddenly jumps into period orbit again with a new cycle. It is clear that these cyclic routes have self-similar bifurcation structure. Furthermore, there are several tangential bifurcation routes in every chaotic region, which lead to generate periodic windows. The periodic window plays an important role in the evolution of dynamical behavior of system (2).

The chaotic regions of system (2) are $[2, 2.035]$, $[2.15, 2.26]$, $[2.4, 2.53]$, $[2.7, 2.855]$, $[3.06, 3.25]$, $[3.505, 3.755]$, $[4.065, 4.395]$, $[4.795, 5.28]$, $[5.76, 6.48]$, $[7.16, 8.36]$, $[9.36, 11.96]$, and $[13.16, 30]$, in which the system is chaotic with a positive Lyapunov exponent $LE_1$ and a zero valued exponent as well as two negative Lyapunov exponents. In the maximum chaotic region $[13.16, 30]$, the system almost locates in the chaotic range, while four main period-windows appear for the regions $[14.11, 14.19]$, $[20.33, 20.79]$, $[25.13, 25.26]$, $[28.99, 29.14]$, respectively. Several typical periodic and chaotic orbits of system (2) are obtained from simulations, as shown in Fig. 4. Figs. 4 (a)–4 (c) show the typical portrait phases of the corresponding three chaotic regions. Figs. 4 (d) to (f) illustrate the typical periodic orbits.
4. Circuit Implementation of the Cubic System

Fig. 5 shows the circuit implementation of the proposed cubic chaotic system, where operational amplifiers can be realized by LM741 or LF347, etc., and multipliers by AD633. The circuit consists of four channels which realize the four state variables $x$, $y$, $z$, and $w$, respectively, and its partial circuit upper the fourth channel is designed to realize the chaotic Lorenz system. With circuit parameters shown in Fig. 5, Lorenz system circuit can generate chaotic signal output. The circuit in the fourth channel is a nonlinear control circuit which will make the whole system exhibit novel chaotic observation.

According to this circuit, we have:

$$a = \frac{R_3}{R_2R_4C_1}, \quad b = \frac{R_5}{R_6R_7C_3}, \quad c = \frac{R_8}{R_7R_9C_2}, \quad d = \frac{R_{10}}{R_{13}R_{14}C_4}. \quad (5)$$

By adjusting the value of resistor $R_{21}$, the phase portraits of chaotic attractor and periodic orbit can be obtained by PSpice simulations, as shown in Fig. 6 (a) to (f), where the initial capacitor voltage of both $C_1$ and $C_2$ are 1 V, and both $C_3$ and $C_4$ are 0 V. It is clear that a good agreement between the numerical simulation and the experimental measurement is observed.

5. Conclusions

In this paper, we have proposed a new chaotic system evolved from Lorenz system. By introducing a nonlinear state feedback controller, which contains three state variables, into the third equation, we obtain the four-dimensional autonomous system with a cubic nonlinearity term. The new cubic system exhibits more complex dynamical behaviors and can generate a complicated 2-scroll chaotic attractor under different parameter conditions. With the variation of parameters, this system can be periodic and chaotic circularly, its cycle width increases with control parameter. Some dynamical properties of this system have been verified by numerical computation and electronic circuit experiment. It is believed that this chaotic circuit will find significant applications in practice.
References


Xiao-Hua Qiao was born in Anhui Province, China, in 1960. He received the B.S. degree from Huaihe Coal Industry Teachers College, Huaihe, in 1978. Now he works with Jiangsu Teachers University of Technology, Changzhou, China. His research interests include power electronics, analysis and simulation in nonlinear systems.