Passivity and Passification of Uncertain Fuzzy Systems with Impulse

Bai-Hua Li and Qi-Shui Zhong

Abstract—The passivity and feedback passification problems of fuzzy systems with parameter uncertainties and impulse are first presented in this paper. Based on the parallel distributed compensation (PDC) technique, some passivity and passification conditions are proposed in terms of linear matrix inequalities (LMIs). Numerical examples are given to show the correctness and effectiveness of our theoretical results.

Index Terms—Fuzzy rules, fuzzy systems, impulsive system, parameter uncertainty, passivity.

1. Introduction

During the last decade, fuzzy systems based on the Takagi-Sugeno (T-S) model\textsuperscript{[1]} have attracted great interest from the scientific and engineering communities. The T-S fuzzy dynamic model is a system described by fuzzy IF-THEN rules which give local linear representations of the underlying nonlinear system\textsuperscript{[1],[2]}. T-S fuzzy systems are usually used to approximate complex nonlinear systems\textsuperscript{[3]}. Generally speaking, uncertainties exist in modeling procedures. Moreover, identification of system parameters is also a hard work and it may lead to errors between actual values of system parameters in an original system and those in a mathematical model. Thus, it is necessary to consider parameters uncertainties for T-S fuzzy systems. Recently, stability and control of T-S fuzzy systems with or without parameter uncertainties have been extensively studied\textsuperscript{[4],[12]}. At the same time, the qualitative properties in the mathematical theory for impulsive systems of differential equations have been intensively developed\textsuperscript{[13],[18]}. It was pointed out that impulsive control can give better performance than continuous control in some practical cases. For example, a central bank can not change its interest rate everyday in order to regulate the money supply in financial market.

On the other hand, the passivity theory, intimately related to circuit analysis methods, has received a lot of attention from the control community during last several decades\textsuperscript{[11],[12],[19],[20]}. The passivity theory provides a nice tool for analyzing the stability of systems, and has found applications in diverse areas such as stability, complexity, signal processing, and nuclear power plant\textsuperscript{[21]}. In [11], the passivity and feedback passification of T-S fuzzy systems with parameter uncertainties were considered. In [12], the passivity problem of uncertain fuzzy systems with time delay was studied. As far as the author knows, unfortunately, no results on this topic have been published with respect to uncertain fuzzy systems with impulse. The proposed work in this paper on the problems of passivity and passification for uncertain fuzzy systems with impulse is new in the literature. The related problems are interesting and challenging.

This paper is organized as follows. In Section 2, the T-S fuzzy model with parameter uncertainties and impulse is first formulated and some preliminaries are introduced. Section 3 derives some passivity criteria for the uncertain fuzzy system with impulse. The passification problem via parallel distributed compensation (PDC) technique is considered in Section 4. The examples are discussed in Section 5 to illustrate the efficiency of the obtained results.

2. Problem Formation and Preliminaries

Throughout this paper, $A>0$ ($<0$) means $A$ is a symmetrical positive (negative) definite matrix, $A^{-1}$ and $A'$ denote the inverse and the transpose of matrix $A$, $A'$ represents the sum of $A$ and $A'$, $I$ is the identity matrix.

Consider the following T-S fuzzy system with parameter uncertainties and impulse, the $i$th rule of this system is described by

\[d\]

\[
\dot{x}(t) = (A_i + \Delta A_i) x(t) + C_i w(t), \quad \forall t \geq 0, \quad t \neq t_k
\]

\[
y(t) = D_i x(t)
\]

\[
\Delta x(t) = d_k x(t), \quad t = t_k, \quad k \in \mathbb{N}
\]
where \( i = 1,2,\ldots,r \) and \( r \) is the number of rules of the T-S fuzzy system. \( z_i(t), z_i'(t), \ldots, z_i^{(r)}(t) \) are the premise variables and each \( M_{ij}(j = 1,2,\ldots,p) \) is a fuzzy set. \( x(t) \in \mathbb{R}^n \) is the state vector, \( w(t) \in \mathbb{R}^q \) is the square-integrable exogenous input and \( y(t) \in \mathbb{R}^q \) is the output vector. \( A_i, C_i, \) and \( D_i \) are some constant matrices with appropriate dimensions. \( \Delta A_i \) is called admissible uncertainties satisfying \( \Delta A_i = NF_i(t)E_i \), where \( N \) and \( E_i \) are known constant matrices and \( F(t) \) is an unknown matrix with the property:

\[
F^T(t)F(t) \leq I, \quad \Delta x(t) = x(t^+) - x(t^-) \quad x(t^+) = \lim_{t \to t^+} x(t),
\]

and \( d_i \) are constants.

With a center-average defuzzifier, the overall uncertain fuzzy system with impulse is represented as

\[
x(t) = \sum_{j=1}^{r} h_i(t) \{ (A_i + \Delta A_i)x(t) + C_i w(t) \}, \quad \forall t \geq 0, \quad t \neq t_i
\]

\[
y(t) = \sum_{j=1}^{r} h_i(t)D_i x(t)
\]

\[
\Delta x(t) = d_i x(t), \quad t = t_i, \quad k \in \mathbb{N}
\]

(1)

where

\[
h_i(t) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}, \quad w_i(z(t)) = \sum_{j=1}^{p} M_{ij}(z_j(t))
\]

and \( M_{ij}(z_j(t)) \) is the grade of membership of \( z_j(t) \) in \( M_{ij} \). Of course, \( h_i(t) \geq 0 \) and \( \sum_{j=1}^{r} h_i(t) = 1 \).

We define some notations which will be used throughout this paper as follows.

\[
\bar{A}_i = A_i + \Delta A_i, \quad \bar{B}_i = B_i + \Delta B_i, \quad \bar{A} = \sum_{i=1}^{r} h_i(t) \bar{A}_i
\]

\[
\bar{B} = \sum_{i=1}^{r} h_i(t) \bar{B}_i, \quad \bar{C} = \sum_{i=1}^{r} h_i(t) C_i
\]

\[
\bar{D} = \sum_{i=1}^{r} h_i(t) D_i, \quad \bar{E} = \sum_{i=1}^{r} h_i(t) E_i
\]

For simplicity, let \( h_i(t) = h_i, \quad x(t) = x, \quad w(t) = w, \quad y(t) = y, \quad u(t) = u, \) and \( V(t, x(t)) = V \). System (1) is equivalent to

\[
\begin{align*}
\frac{d}{dt} x &= \bar{A} x + \bar{C} w, \quad \forall t \geq 0, \quad t \neq t_i \\
y &= \bar{D} x \\
\Delta x &= d_i x, \quad t = t_i, \quad k \in \mathbb{N}
\end{align*}
\]

(2)

This paper adopts the following widely accepted definition of passivity.

**Definition 1.** System (2) is called passive if there exists a scalar \( \gamma \geq 0 \) such that

\[
2 \int_0^t w^T(s)y(s)ds \geq -\gamma \int_0^t w^T(s)x(s)ds
\]

(3)

for all \( t_p \geq 0 \) and for all solution of (2) under zero initial condition.

### 3. Passivity of Uncertain Fuzzy Systems with Impulse

In this Section, we will derive some passivity criteria for the uncertain fuzzy system with impulse (2).

**Theorem 1.** If there exist a matrix \( P > 0 \), scalar \( \varepsilon_i > 0 \) \( (i = 1,2,\ldots,r) \) and \( \gamma \geq 0 \) such that the following inequalities hold:

\[
\begin{bmatrix}
 PA_i + \varepsilon_i E_i^T E_i & * \\
 C_i^T P - D_i & -\gamma I \\
 N^T P & 0 & -\varepsilon_i I
\end{bmatrix} < 0, \quad i = 1,2,\ldots,r
\]

(5)

where \( * \) denotes the transposed element in the symmetric position, then the uncertain fuzzy system with impulse (2) is passive in the sense of Definition 1.

**Proof.** Choose Lyapunov function candidate as

\[
V[t, x(t)] = x^T P x.
\]

We have

\[
\begin{align*}
\frac{d}{dt} V &= -2y^T w - \gamma w^T w \\
&= (d \frac{dx}{dt})^T P x + x^T P (d \frac{dx}{dt}) - (x^T D^T w + w^T D x) - \gamma w^T w \\
&= x^T (A^T P + PA) x + x^T (PC - D^T) w + w^T (C^T P - D) x + w^T (-\gamma I) w \\
&= x^T \begin{bmatrix}
 PA & * \\
 C^T P - D & -\gamma I
\end{bmatrix} x.
\end{align*}
\]

(7)

Define \( \eta^T \Omega \eta < 0, \quad i = 1,2,\ldots,r \).

Recalling that \( F^T(t)F(t) \leq I \), one can obtain the following inequality:

\[
\varepsilon_i [F(t)E_i a]^T [F(t)E_i a] \leq \varepsilon_i a^T E_i^T E_i a.
\]

(9)

From (8) and (9),

\[
\bar{\eta}^T \Theta \bar{\eta} \leq \eta^T \Omega \eta < 0, \quad i = 1,2,\ldots,r
\]

(10)

where

\[
\bar{\eta} = [a^T, b^T]^T, \quad \Theta_i = \begin{bmatrix}
 PA & * \\
 C_i^T P - D_i & -\gamma I
\end{bmatrix}.
\]
Since \( a \) and \( b \) are arbitrary vectors, we have \( \Theta_i < 0 \), \( i = 1, 2, \cdots, r \). Consequently,

\[
\sum_{i=1}^{r} h_i \Theta_i \leq \sum_{i=1}^{r} h_i \begin{bmatrix} \overrightarrow{PA} & \ast \\ \overrightarrow{C}'P - D - \gamma I \end{bmatrix} < 0
\]

According to (7) and (11), one has

\[
\frac{d}{dt} V = -2y^T w - \gamma w^T w < 0 \tag{12}
\]

By integrating (12) with respect to \( t \) over the time period \((0, t_p)\), it yields that

\[
-2 \int_0^{t_p} w^T y dt - \gamma \int_0^{t_p} w^T w dt \leq -\int_0^{t_p} \frac{d}{dt} V dt \tag{13}
\]

According to the definition of \( V \), we have \( V(0) = 0 \), \( V(t_p) > 0 \), and notice that \( -2 \leq d_i \leq 0 \), when \( t_p \in (t_i, t_{i+1}] \), we have

\[
\int_0^{t_p} \frac{d}{dt} V dt = \int_0^{t_i} \frac{d}{dt} V dt + \int_{t_i}^{t_{i+1}} \frac{d}{dt} V dt + \cdots + \int_{t_{i-1}}^{t_i} \frac{d}{dt} V dt + \int_{t_{i-1}}^{t_i} \frac{d}{dt} V dt + \cdots + \int_0^{t_i} \frac{d}{dt} V dt
\]

\[
= V(t_i) - V(0) + V(t_{i+1}) - V(t_i) + \cdots + V(t_p) - V(t_{i+1}) + V(t_{i+1}) - V(t_p)
\]

\[
= -\sum_{i=1}^{r} d_i (2 + d_i) V(t_i) + V(t_p) \geq 0 \tag{14}
\]

From (13) and (14), (3) holds, and hence the uncertain fuzzy system with impulsive (1) is passive in the sense of Definition 1.

For the nominal system of (1), i.e., with \( \Delta A_i = 0 \), similar to the proof of Theorem 1, we have Corollary 1.

**Corollary 1.** If there exist a matrix \( P > 0 \) and a scalar \( \gamma \geq 0 \) such that the following inequalities hold:

\[
-2 \leq d_k \leq 0, \; k \in \mathbb{N} \tag{15}
\]

\[
\begin{bmatrix} \overrightarrow{PA} & \ast \\ \overrightarrow{C}'P - D - \gamma I \end{bmatrix} < 0, \; i = 1, 2, \cdots, r \tag{16}
\]

then the system (1) with \( \Delta A_i = 0 \) is passive in the sense of Definition 1.

4. State Feedback Passification

In this section, we design a state feedback controller to make the closed-loop fuzzy system with impulsive passive. We consider the uncertain impulsive fuzzy system (1) with the control input of the following form.

**Rule 1:** If \( z_1(t) \) is \( M_{i1} \), \( z_2(t) \) is \( M_{i2} \), \( \cdots \), and \( z_p(t) \) is \( M_{ip} \), then

\[
\frac{d}{dt} x = (A_i + \Delta A_i)x + (B_i + \Delta B_i)u + C_i w, \; \forall t \geq 0, \; t \not= t_k
\]

\[
y = D x + E_i u
\]

\[
\Delta x = d_i x, \; t = t_k, \; k \in \mathbb{N} \tag{17}
\]

where \( u \in \mathbb{R}^m \) is the control input, \( \Delta B_i \) is called admissible uncertainties satisfying \( \Delta B_i = NF(t)E_{2i} \). Based on the concept of PDC(3), we consider the following fuzzy control law for (1) with control input of the following form.

**Control Rule i:** If \( z_1(t) \) is \( M_{i1} \), \( z_2(t) \) is \( M_{i2} \), \( \cdots \), and \( z_p(t) \) is \( M_{ip} \), then \( u = -\overrightarrow{K_i} x \), where \( \overrightarrow{K_i} \) are constant gain matrices to be determined later. The overall controller is represented as

\[
u = -\sum_{i=1}^{r} h_i \overrightarrow{K_i} x \tag{18}
\]

Applying (18) to (17), the resulting closed-loop system can be recasted as

\[
\frac{d}{dt} x = (A - B \sum_{i=1}^{r} h_i \overrightarrow{K_i}) x + C w, \; \forall t \geq 0, \; t \not= t_k
\]

\[
y = (D - E \sum_{i=1}^{r} h_i \overrightarrow{K_i}) x
\]

\[
\Delta x = d_i x, \; t = t_k, \; k \in \mathbb{N} \tag{19}
\]

where

\[
W = A - B \sum_{i=1}^{r} h_i \overrightarrow{K_i} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j (\overrightarrow{A_i} - \overrightarrow{B_i} \overrightarrow{K_j})
\]

**Theorem 2.** If there exist matrices \( P > 0 \), \( R > 0 \), \( K_j \) \( (j = 1, 2, \cdots, r) \), scalar \( e_0 > 0 \), \( (i, j = 1, 2, \cdots, r) \), and \( \gamma \geq 0 \) such that the following inequalities hold:

\[
-2 \leq d_k \leq 0, \; k \in \mathbb{N} \tag{20}
\]

and for \( i, j = 1, 2, \cdots, r \),

\[
\Gamma_i = \begin{bmatrix} A - B K_j + e_0 NN^T & * & * \\ Q_1 & Q_2 & * \\ C_i^T - D R - E_i K_j & C_i^T - \gamma I & * \\ E_{ij} R - E_{ij} K_j & 0 & 0 & -e_j I \end{bmatrix} < 0 \tag{21}
\]

where

\[
Q_1 = P - R - A R - B K_j + e_0 NN^T
\]

\[
Q_2 = -2R + e_0 NN^T
\]

Then the closed-loop system (19) is passive in the sense of Definition 1 with the control gain matrix:

\[
K_j = K_j R_i^{-1}
\]

**Proof.** Let \( R = R_i^{-1} \), \( P = R P R_i \). Choose Lyapunov function candidate as

\[
V = x^T P x \tag{22}
\]
Note that (19) is equivalent to
\[
\frac{d}{dt} x = f, \quad f = Wx + Cw, \quad \forall t \geq 0, \quad t \neq t_k
\]
\[
y = (D - E \sum_{j=1}^{r} h_j K_j)x
\]
\[
\Delta x = d_j x, \quad t = t_k, \quad k \in \mathbb{N}.
\]
And then the derivative of \( V \) is
\[
\frac{d}{dt} V = 2 x^T \dot{P} \frac{d}{dt} x = 2 x^T \dot{P} f
\]
\[
= 2 x^T \dot{P} f + 2 (x^T + f^T) \dot{R}(Wx + Cw - f)
\]
\[
= 2 x^T \dot{P} f + 2 x^T \dot{R}Wx + 2 x^T \dot{R}Cw
\]
\[
- 2 x^T \dot{R} f + 2 f^T \dot{R}w + 2 f^T \dot{R}Cw - 2 f^T \dot{R} f
\]
\[
= \left[ x^T \right] \begin{bmatrix} \dot{R}W & * & * \\ \dot{P} & - \dot{R} & * & * \\ C^T \dot{R} & C^T \dot{R} & 0 & w \end{bmatrix} x
\]
On the other hand, we have
\[
-2y^T w = \sum_{i=1}^{r} h_i h_i \{x^T [-D_i, E_i] \} y
\]
+ \( w^T [-D_i, E_i] x \).

Then
\[
\frac{d}{dt} V - 2 y^T w - \gamma w^T w = \sum_{i=1}^{r} h_i h_i \dot{M}_y \theta
\]
where
\[
\theta = [x^T, f^T, \dot{w}^T]^T
\]
\[
\dot{M}_y = \begin{bmatrix} \dot{R}(\bar{A}, - \bar{B}, K_j) & * & * \\ \dot{P} - \dot{R} + \dot{R}(\bar{A}, - \bar{B}, K_j) & -2R & * \\ C^T \dot{R} - D_i, E_i, K_j & C^T \dot{R} - \gamma I \end{bmatrix}
\]
Define \( \xi = [a^T, b^T, c^T, (a^T + b^T) NF(t)]^T \) with arbitrary vectors \( a, b \) and \( c \) of appropriate dimension. According to (21),
\[
\xi^T \Gamma \xi, \xi^T \xi, \xi < 0, \quad i, j = 1, 2, \ldots, r.
\]
Recalling that \( F^T(t) F(t) \leq I \), one can obtain the following inequality:
\[
\xi^T \dot{M}_y \xi, \xi^T \xi, \xi < 0, \quad i, j = 1, 2, \ldots, r
\]
where
\[
\xi = [a^T, b^T, c^T]^T
\]
Since \( a, b \) and \( c \) are arbitrary vectors, we have \( \dot{M}_y < 0 \), \( i, j = 1, 2, \ldots, r \). Let \( \Sigma = diag[R, R, I] \), then
\[
\dot{M}_y = \Sigma \dot{M}_y \Sigma < 0.
\]
From (23) and (27), one can obtain
\[
\frac{d}{dt} V - 2 y^T w - \gamma w^T w < 0.
\]
The rest of the proof is similar to Theorem 1 and therefore omitted for simplicity.

Similar to Corollary 1, for the closed-loop fuzzy system (19) without parameter uncertainties, we have the following corollary.

**Corollary 2.** If there exist matrices \( P > 0, \ R > 0, \ K_j \), \( j = 1, 2, \ldots, r \), and \( \gamma \geq 0 \) such that the following inequalities hold:
\[
-2 \leq d_k \leq 0, \quad k \in \mathbb{N}
\]
and for \( i, j = 1, 2, \ldots, r \),
\[
\begin{bmatrix} \bar{A}, \bar{B}, K_j & * & * \\ \bar{P} - \bar{R} + \bar{A}, \bar{B}, K_j & -2R & * \\ C^T \bar{R} - D_i, E_i, K_j & C^T \bar{R} - \gamma I \end{bmatrix} < 0.
\]
Then the closed-loop system (19) with \( \Delta A_j = 0 \) and \( \Delta B_j = 0 \) is passive in the sense of Definition 1 with the control gain matrix \( K_j = K \cdot R^{-1} \).

5. **Numerical Examples**

To demonstrate the effectiveness of our theoretical results, consider the following fuzzy system with impulse:
\[
\begin{aligned}
\frac{d}{dt} x(t) &= \sum_{i=1}^{2} \frac{d}{dt} h_i(t) \{ (A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) \} \\
&\quad + C_i w(t), \quad \forall t \geq 0, \quad t \neq t_k
\end{aligned}
\]
\[
y(t) = \sum_{i=1}^{2} h_i(t) (D_i x(t) + E_i u(t))
\]
\[
\Delta x(t) = d_j x(t), \quad t = t_k, \quad k \in \mathbb{N}
\]
where \( x \in \mathbb{R}^2, \ u \in \mathbb{R}^2, \ w \in \mathbb{R}^2, \ \Delta A_i = \Delta B_i = NF(t)E \), and
\[
\begin{align*}
A_1 &= \begin{bmatrix} -0.3 & 0.2 \\
0 & -0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.2 & 0.1 \\
0.4 & -0.2 \end{bmatrix} \\
B_1 &= \begin{bmatrix} 0.2 & 0.4 \\
0 & -0.8 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3 & 1 \\
0.7 & -0.3 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix}
\end{align*}
\]
D_1 = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.2 & 0.3 \end{bmatrix}

E_1 = \begin{bmatrix} 0.2 & -0.1 \\ 0.2 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.4 \end{bmatrix}

N = \begin{bmatrix} -0.2 & 0 \\ 0.3 & -0.1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.3 & 0.5 \\ 0.1 & 1 \end{bmatrix}

F(t) = \begin{bmatrix} \cos t & 0 \\ 0 & \sin t \end{bmatrix}, \quad d_k = -1

First, we consider this system without control input. Applying Theorem 1 and using the MATLAB LMI toolbox, we can obtain

\[ P = \begin{bmatrix} 6.8066 & -4.1869 \\ -4.1869 & 11.7333 \end{bmatrix}, \quad e_1 = 5.6364, \quad e_2 = 18.6252, \quad \gamma = 216.4484 \]

which means that this system without control input is passive in the sense of Definition 1.

Next, we consider the closed-loop system. Applying Theorem 2, we can obtain

\[ P = \begin{bmatrix} 2.6344 & -0.5424 \\ -0.5424 & 2.0904 \end{bmatrix}, \quad R = \begin{bmatrix} 1.6268 & -0.2593 \\ -0.2593 & 1.1582 \end{bmatrix} \]

\[ K_1 = K_2 = \begin{bmatrix} 1.7391 & 1.3545 \\ 0.9083 & -1.8348 \end{bmatrix}, \quad e_{11} = e_{12} = 2.8901, \quad e_{21} = e_{22} = 2.8995, \quad \text{and} \quad \gamma = 5. \]

Subsequently we can obtain the feedback gain

\[ \hat{K}_1 = \hat{K}_2 = \begin{bmatrix} 1.3019 & 1.4610 \\ 0.3172 & -1.5132 \end{bmatrix} \]

6. Conclusions

Passivity and state feedback passification of uncertain fuzzy systems with impulse have been studied, and the corresponding conditions for passivity and passification were expressed in terms of LMIs. Numerical examples have been presented to demonstrate the effectiveness and correctness of the theoretical results. It is believed that the problem described in this paper is a maiden research topic, so we hope it could arouse readers’ curiosity towards such systems.

References


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