Numerical Solution for Fractional-Order Differential Systems with Time-Domain and Frequency-Domain Methods

Ke Xiao, Shang-Bo Zhou, and Wei-Wei Zhang

Abstract—For a general nonlinear fractional-order differential equation, the numerical solution is a good way to approximate the trajectory of such systems. In this paper, a novel algorithm for numerical solution of fractional-order differential equations based on the definition of Grunwald–Letnikov is presented. The results of numerical solution by using the novel method and the frequency-domain method are compared, and the limitations of frequency-domain method are discussed.

Index Terms—Analytical solution; frequency domain; fractional order; numerical solution; time domain.

1. Introduction

The idea of fractional-order integrals and derivatives has been known since the development of the regular calculus. In fact, fractional calculus may date from the 17th century[1][2]. In recent years, considerable attention has been paid to finding chaotic behaviors in fractional-order models, for instance, the fractional-order Duffing system[3][4], the fractional-order Chua’s system[5][6], the fractional-order Chen’s system[7][8], the fractional-order Lu system[9][10], the fractional-order Rossler system[11], the fractional-order jerk model[12], the fractional-order cellular neural network[13][14], and the fractional-order neural network[15] are well known and can exhibit chaotic behavior. Consequently it is meaningful to investigate the nonlinear fractional-order systems. For a general nonlinear fractional-order system, it is difficult to find out the analytical solution, so in most of literature in the fractional-order chaos field, approximation and numerical techniques must be used to simulate the chaotic behavior. In other words, numerical solution is an important measure to analyze the nonlinear fractional-order equations.

The methods of numerical solution can be divided into two different kinds; one is called “time-domain method”, and the other is called “frequency-domain method”. The former kind of methods is simulating a fractional-order system by numerically solving its fractional-order differential equations. In this paper, a novel time-domain method for numerical solution is introduced. The presented method can be not only applied to linear equations, but also applied to nonlinear equations. The latter is simulating the fractional-order system based on the frequency-domain approximations. Firstly the fractional-order equation is considered in the frequency-domain, and then the Laplace form of the fractional integral operator is replaced by its integer order approximated expressions. Then the approximated expressions in the frequency-domain are transformed into the time-domain. Lastly, the ordinary differential equation can be solved by applying well known numerical methods, such as Runge-Kutta algorithm. The frequency-domain method is widely used in many literatures to simulate the chaotic behavior of the fractional-order systems.

There are many definitions of fractional derivatives[1]. Grunwald-Letnikov (GL) definition[16][17] of non-integer integration and differentiation is given by

\[ \left(\frac{D^\alpha}{dt^\alpha}\right) f(t) = \lim_{h \to 0} \sum_{j=0}^{n} (-1)^{j} \binom{\alpha}{j} f(t - jh) \]

where \( \binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} \), and \( a \) and \( t \) are the limits of the operation, and \( h \) is the time step.

This formula can be reduced to

\[ \left(\frac{D^\alpha}{dt^\alpha}\right) y(t_n) = h^{-\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} y_{n-j} \]

where

\[ \omega_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}, \ (j = 0, 1, 2, \cdots) \]

\[ \binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} \]

and \( h \) is the time step.

The rest of the paper is organized as follows. In Section 2, the methods and numerical simulation of a fractional-order are described and analyzed. The results of the presented time-domain method are compared with that of frequency-domain method with two examples, and some limitations of frequency-domain method are presented in section 3. Section 4 gives the conclusions.
2. Methods and Numerical Simulation of a Fractional-Order System

2.1 Time-Domain Method

By using (2), we can construct an algorithm to solve fractional-order differential equations. Let us consider the following general nonlinear system

\[ D_\alpha^a y(t) + N(y(t)) = g(t), \quad \alpha > 0 \] (3)

where \( N \) represents a nonlinear operator which include \( y(t) \), \( g(t) \) is a function with respect to \( t \).

From (2) and (3), (3) is transformed as follows:

\[ h^{-\alpha} \sum_{j=0}^{m} a_j y_{m-j} + N(y(m)) = g(t_m), \]

where \( m = 1, 2, \ldots \lfloor \frac{1-a}{h} \rfloor \) is a function with respect to \( t \).

Equation (4) can be transformed to

\[ y(m) = h^{-\alpha} (g(t_m) - N(y(m))) - \sum_{j=1}^{m} a_j y_{m-j}, \hspace{1cm} m = 1, 2, \ldots \lfloor \frac{1-a}{h} \rfloor \] (5)

For (5), general iterative method can be utilized to solve \( y_m \). If the equation is convergent, (5) can be constructed as

\[ y(m) = h^{-\alpha} (g(t_m) - N(y(m))) - \sum_{j=1}^{m} a_j y_{m-j}, \hspace{1cm} m = 1, 2, \ldots \] (6)

The frequency-domain method is based on the Laplace form of the fractional integral operator.

2.2 Frequency-Domain Method

The frequency-domain method is based on the approximation of the fractional-order system behavior in the frequency. Before using the frequency-domain method to solve a fractional-order equation, the transfer function approximations of \( 1/s^\alpha \) must be obtained. In [6], [18], [19], the algorithm has been proposed for calculating transfer function approximations of \( 1/s^\alpha \).

The approximate linear transfer functions for the fractional integrator of order that varies from 0.1 to 0.9 with the maximum discrepancy of 2 dB from \( \omega=10^{-2} \) to \( 10^2 \text{ rad/s} \) are given in Table 1 of [6].

To simulate a fractional-order system by using frequency-domain method, firstly, the fractional-order equation is considered in the frequency-domain and then the Laplace form of the fractional integral operator is replaced by its integer order approximated expressions. Then the approximated expressions in the frequency-domain are transformed into the time-domain. Lastly, the ordinary differential equation can be solved by applying well known numerical methods, such as Runge-Kutta algorithm.

The frequency-domain method is widely utilized in many literatures to simulate the chaotic behavior of the fractional-order systems [5]-[12]. The chaotic behavior of fractional-order Rössler system is displayed by using frequency-domain method in Fig. 1 of [11].

3. Numerical Solution Analyzing

By investigating and simulating the one-scroll attractor of the fractional-order Rössler system, the availability of time-domain method and the frequency-domain method are illuminated for nonlinear fractional-order differential equations. Some limitations of frequency-domain method are indicated after investigations. The two simple examples are given and solved by using two methods, and the numerical solutions are compared with analytical solutions.

3.1 Models

In order to compare conveniently, a half order differential equations are considered here [20].

A. Model 1

The first model of the fractional-order differential system is as follows:

\[ D_1^{\frac{1}{2}} y(t) - t^\frac{1}{2} = 0, \hspace{1cm} 0 \leq t \leq 1 \] (8)

The analytical solution of (8) is

\[ y(t) = \frac{2}{315 \sqrt{\pi}} x^2 \]

The numerical solution and the error by using frequency-domain method are given in Fig. 2, and the
The numerical solution and the error by using time-domain method are given in Fig. 3.

![Fig. 3](image1)

Fig. 3. Waveform diagram of solutions and error: (a) analytical solution and numerical solution by time-domain method; (b) the error $e(t)$ between the analytical solution and numerical solution. (time step $h=0.001$).

The numerical solution and the error by using frequency-domain method are given in Fig. 4, and the numerical solution and the error by using time-domain method are given in Fig. 5.

![Fig. 4](image2)

Fig. 4. Waveform diagram of solutions and error: (a) analytical solution and numerical solution by frequency-domain method; (b) the error $e(t)$ between the analytical solution and numerical solution. (time step $h=0.001$).

**B. Model 2**

The second model of the fractional-order differential system is as follows:

$$D^{0.5} y(t) + D^{0.5} y(t) = \frac{\sqrt{t}}{(1+t)^{2}}, \quad 0 \leq t \leq 1 \quad (9)$$

The analytical solution of (9) is $y(t) = 2 \arcsin h(\sqrt{t})/\sqrt{(1+t)}$. 

![Fig. 5](image3)

Fig. 5. Waveform diagram of solutions and error: (a) analytical solution and numerical solution by time-domain method; (b) the error $e(t)$ between the analytical solution and numerical solution. (time step $h=0.001$).
3.2 Analyze the Numerical Solution

The numerical solutions obtained by frequency-domain method and the presented time-domain method all approximate the analytical solution well for some systems. As using the frequency-domain method, the equations need to take Laplace transform and inverse Laplace transform, but sometimes it is difficult to take Laplace transform or inverse Laplace transform if the system equation includes variable $t$ explicitly, so this method can be more complex. When Runge-Kutta algorithm is used to solve the integer high order approximated equation, the initial conditions of this equation are hard to be chosen. These problems will not exist in the presented time-domain method, and the initial condition considers only $y(0)$.

As using the frequency-domain method to solve $m$ ($0<m<1$) order differential equation, the equation must satisfy that the solution $y(t)$ defined on the interval $[a, b]$ have continuous $(N−1)$ order differential equation (the value of $N$ is given in Table 1 and Table 2 of [19]); otherwise, we cannot obtain the numerical solution at this point. For (9), the Fig. 4 (a) shows the numerical solution by using frequency-domain method, the numerical solution causes large undesired error since $y(t)$ cannot derivative at the point $t=0$. Fig. 2 (b) and Fig. 4 (b) show the error $e(t)$ between the analytical solution and numerical solution by using frequency-domain method. For (8), the maximal error of numerical solution is $e(t)=3.7\times10^{-2}$; for (9), the maximal error of numerical solution is $e(t)=0.7$. Fig. 3 (b) and Fig. 5 (b) show the error $e(t)$ between the analytical solution and numerical solution by using the presented time-domain method, for (8), the maximal error of numerical solution is $e(t)=5.1\times10^{-4}$; for (9), the maximal error of numerical solution is $e(t)=7.3\times10^{-3}$. Thereby, it can obviously see that the numerical solution by using time-domain method is more accurate than that by using frequency-domain method.

Due to the frequency-domain method is a solution that translates fraction-order into time-domain integer high order differential equation, then the total order of system is not equal to the highest derivative of the fractional-order differential equation, and in some way the fractional-order of system cannot be replaced by integral-order without real reason; the fractional-order systems have an unlimited memory, but the integral-order systems have a limited memory, hence, it obviously changes the character of fractional-order. However, the time-domain method gets the solution by using fraction-order definition directly, so the presented time-domain method keeps the character of fraction-order well, and it can get different precise numerical solution by setting time step small enough. At the same time, the presented time-domain method can solve any order differential equation conveniently, and it needs not to deduce approximate expressions of the integer order, and therefore the time-domain method is obviously better than frequency-domain method in this aspect.

4. Conclusions

It is well known that the frequency-domain method is widely utilized to simulate and investigate the chaotic behavior of the nonlinear systems. However, some limitations given in section 3 should be considered in the use of this method. In this paper, a novel time-domain method for numerical solution of the nonlinear fractional-order differential equations based on the definition of Grunwald–Letnikov is presented. The numerical solution by using the presented time-domain method also approximates the analytical solution. Compared with the frequency-domain method, the presented method can conquer the limitations of the frequency-domain method which we discuss above. The advantages of the time-domain method are that the order can be set randomly, computation and program process are much simpler, and the numerical solution is more approximate to the analytical solution. However, the use of the presented time-domain method in more complex nonlinear fractional-order system needs to do research further.
References


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