Delay-Dependent Exponential Stability Criterion for BAM Neural Networks with Time-Varying Delays

Wei-Wei Su and Yi-Ming Chen

Abstract—By employing the Lyapunov stability theory and linear matrix inequality (LMI) technique, delay-dependent stability criterion is derived to ensure the exponential stability of bi-directional associative memory (BAM) neural networks with time-varying delays. The proposed condition can be checked easily by LMI control toolbox in Matlab. A numerical example is given to demonstrate the effectiveness of our results.

Index Terms—Bi-directional associative memory (BAM) neural networks, delay-dependent exponential stability, linear matrix inequality (LMI), lyapunov stability theory, time-varying delays.

1. Introduction

The bi-directional associative memory (BAM) neural networks were first proposed and researched by Kosko[1],[2] for their useful memory application in pattern recognition, solving optimization problems and automatic connection engineering. Hence, the BAM neural networks have been one of the most interesting topics and have attracted the attention of many researchers.

Time delay will inevitably occur in electronic neural networks owing to the unavoidable finite switching speed of amplifiers. The delay is a source of instability and oscillatory response of networks, so it is more in accordance with this fact to study the BAM neural networks with constant or time-varying delays. Most of these studies involve various dynamical behaviors, such as, periodic oscillation[3], chaos and bifurcation[4], stability[5]-[10]. In designing neural networks, one concerns not only the stability of the system but also the convergence rate, so it is important to determine the exponential stability and estimate the exponential convergence rate. Thus, global exponential stability for neural networks has been investigated in very recent years[7]-[10]. In [7], the authors give the exponential stability criterion in terms of algebraic inequalities. In [10], the authors study the exponential stability and estimate the exponential convergence rate for BAM neural networks with constant or time-varying delays. It will be less conservative if we choose a more suitable Lyapunov functional.

In our paper, we develop a new delay-dependent exponential stability condition for BAM neural networks with time-varying delays by utilizing Lyapunov functions. The delay-dependent stability criterion is derived as linear matrix inequality (LMI) that can be efficiently solved by the LMI control toolbox in Matlab. The effectiveness of the proposed stability criterion and its improvement over some existing results are illustrated in numerical example.

2. Problem Statement

Consider the following BAM neural networks with time-varying delays:

\[
\begin{align*}
\frac{du(t)}{dt} &= -Au(t) + Wg_1(v(t - \tau(t))) + I \\
\frac{dv(t)}{dt} &= -Bv(t) + Vg_2(u(t - \sigma(t))) + J
\end{align*}
\]

(1)

where

\[
\begin{align*}
u(t) &= [u_1(t), u_2(t), \ldots, u_n(t)]^T \\
v(t) &= [v_1(t), v_2(t), \ldots, v_m(t)]^T \\
A &= \text{diag}(a_1, a_2, \ldots, a_n), \quad a_i > 0 \\
B &= \text{diag}(b_1, b_2, \ldots, b_m), \quad b_j > 0 \\
W &= (w_{ij})_{m \times n}, \quad V = (v_{ji})_{n \times m} \\
I &= [I_1, I_2, \ldots, I_n]^T, \quad J = [J_1, J_2, \ldots, J_m]^T
\end{align*}
\]

where \( u_i \) and \( v_j \) are the activations of the \( i \)-th neurons and \( j \)-th neurons, respectively, \( w_{ij} \) and \( v_{ji} \) are the connection weights, \( I \) and \( J \) denote the external inputs. The delays considered are time-varying, which satisfy:

\[
0 \leq \sigma(t) \leq \sigma, \quad \frac{d\sigma(t)}{dt} \leq \mu_1 < 1
\]

\[
0 \leq \tau(t) \leq \tau, \quad \frac{d\tau(t)}{dt} \leq \mu_2 < 1.
\]

In this paper, it is assumed that the activate functions \( g_{ij}, g_{2j} \) \( (j = 1, 2, \ldots, m; i = 1, 2, \ldots, n) \) are bounded and satisfy the following properties (A and B).
\[(A) \quad 0 \leq \frac{g_{ij}(x) - g_{ij}(y)}{x - y} \leq l_{ij}\]
\[0 \leq \frac{g_{ij}(x) - g_{ij}(y)}{x - y} \leq l_{ij}\]
where \(l_{ij}, l_{ij}\) are real numbers.

It is clear that under the assumption, \((1)\) has at least one equilibrium point. Assume that \(u^* = (u_1^*, u_2^*, \ldots, u_m^*)^T\) and \(v^* = (v_1^*, v_2^*, \ldots, v_m^*)^T\) are the equilibrium points of the system, then we will shift the equilibrium points to the origin by the transformation \(x_i(t) = u_i(t) - u_i^*\) and \(y_j(t) = v_j(t) - v_j^*\).

Then the transformed system is as follows:
\[
\begin{align*}
\frac{dx(t)}{dt} &= -Ax(t) + Wf_1(y(t - \tau(t))) \\
\frac{dy(t)}{dt} &= -By(t) + Vf_2(x(t - \sigma(t)))
\end{align*}
\]
where
\[
f_1(y_j(t)) = g_{ij}(y_j(t) + v_j^*) - g_{ij}(v_j^*)
\]
\[
f_2(x_i(t)) = g_{ij}(x_i(t) + u_i^*) - g_{ij}(u_i^*)
\]

The activate functions \(f_1, f_2\) are bounded and satisfy the following properties:
\[(B) \quad 0 \leq \frac{f_1(x)}{x} \leq l_{ij}, 0 \leq \frac{f_2(x)}{x} \leq l_{ij}\]
where \(f_1(0) = 0\), \(f_2(0) = 0\), for any \(x \in \mathbb{R}\), \(j = 1, 2, \ldots, m; \ i = 1, 2, \ldots, n\).

**Definition 1.** The trivial solution of \((2)\) is said to be globally exponential stable, if there exist constant \(k > 0\), \(K \geq 1\), such that:
\[
\|x(t)\|^2 + \|y(t)\|^2 \leq K (\|\phi\|^2 + \|\psi\|^2) e^{-2kt}
\]
where we denote
\[
\|\phi\|^2 + \|\psi\|^2 = \sup_{[-\max(\sigma, r), 0]} \|x(t)\|^2 + \sup_{[-\max(\sigma, r), 0]} \|y(t)\|^2
\]

**Lemma 1** (S-procedure) Let \(T_i(i = 0, 1, \ldots, n)\). The following statement is true:

If there exist real scalars \(\tau_i \geq 0\), \(i = 1, 2, \ldots, n\), such that
\[T_0 - \sum_{i=1}^n \tau_i T_i < 0\]
then, for any \(x\), satisfying \(x^T T_0 x \leq 0\), \((i = 1, 2, \ldots, n)\), we have \(x^T T_0 x < 0\).

### 3. Main Results

In this section, we discuss global exponential stability of (1). By constructing a new Lyapunov function, we obtain delay-dependent stability criterion in terms of LMIs.

**Theorem 1.** System \((2)\) is global exponentially stable if there exist positive matrices \(P_1, P_2, Q_1, Q_2, R_1, R_2\), positive diagonal matrices \(D, E, U_1, U_2\), such that
\[
\Xi = \begin{bmatrix}
\Phi_1 & 0 \\
-1 - \mu_1 e^{-2k_1} R_1 & 0 \\
\Phi_3 & P_1 W \\
0 & Q_2 - 2U_2 & DW \\
0 & 0 & \Phi_4 \\
0 & 0 & \Psi_1 \\
0 & 0 & \Psi_3 \\
0 & 0 & \Psi_4 \\
\Phi_5 & 2kP_1 - 2P_1 A + R_1 \\
\Phi_6 & -1 - \mu_2 e^{-2k_2} R_2 \\
\Psi_1 & 2kP_2 - 2P_2 B + R_2 \\
\Phi_2 & kW - BE + L_1 U_1 \\
\Phi_3 & -1 - \mu_3 e^{-2k_3} Q_1 \\
\Psi_3 & L_1 - \text{diag}(l_{i1}, l_{i2}, \ldots, l_{in}) \\
\Psi_2 & L_2 - \text{diag}(l_{j1}, l_{j2}, \ldots, l_{jn})
\end{bmatrix} < 0
\]
\[
\Sigma = \begin{bmatrix}
\Phi_1 & 0 \\
-1 - \mu_1 e^{-2k_1} R_1 & 0 \\
\Phi_3 & P_1 W \\
0 & Q_2 - 2U_2 & DW \\
0 & 0 & \Phi_4 \\
0 & 0 & \Psi_1 \\
0 & 0 & \Psi_3 \\
0 & 0 & \Psi_4 \\
\Phi_5 & 2k P_1 - 2P_1 A + R_1 \\
\Phi_6 & -1 - \mu_2 e^{-2k_2} R_2 \\
\Psi_1 & 2k P_2 - 2P_2 B + R_2 \\
\Phi_2 & kW - BE + L_1 U_1 \\
\Phi_3 & -1 - \mu_3 e^{-2k_3} Q_1 \\
\Psi_3 & L_1 - \text{diag}(l_{i1}, l_{i2}, \ldots, l_{in}) \\
\Psi_2 & L_2 - \text{diag}(l_{j1}, l_{j2}, \ldots, l_{jn})
\end{bmatrix} < 0
\]

**Proof.** For system \((2)\), we choose the following Lyapunov-Krasovskii function
\[
V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) + V_6(t) + V_7(t) + V_8(t)
\]
where
\[
V_1(t) = e^{2k_1} x^T(t) P_1 x(t)
\]
\[
V_2(t) = e^{2k_2} y^T(t) P_2 y(t)
\]
\[
V_3(t) = \int_{t - \sigma(t)}^t e^{2k_1} x^T(s) R_1 x(s) ds
\]
\[
V_4(t) = \int_{t - \tau(t)}^t e^{2k_2} y^T(s) R_2 y(s) ds
\]
\[
V_5(t) = \int_{t - \sigma(t)}^t e^{2k_1} f_1^T(x(s)) Q_1 f_1(x(s)) ds
\]
\[
V_6(t) = \int_{t - \tau(t)}^t e^{2k_2} f_2^T(y(s)) Q_2 f_2(y(s)) ds
\]
\[
V_7(t) = 2 \sum_{j=1}^m d_j e^{2k_1} \int_0^{\sigma(t)} f_1_j(s) ds
\]
\[
V_8(t) = 2 \sum_{j=1}^m e_j e^{2k_2} \int_0^{\tau(t)} f_2_j(s) ds
\]
where \(P_1, P_2, Q_1, Q_2, R_1, R_2\) are positive matrices, \(d_j (j = 1, 2, \ldots, m)\) and \(e_j (j = 1, 2, \ldots, m)\) are positive scalars.

Taking the derivative of \(V(t)\) along the trajectory of (2), then we get
\[
\frac{dV_1(t)}{dt} = 2ke^{2\xi}x^T(t)p_1x(t) + 2e^{2\xi}x^T(t)p_1dx(t)
\]
\[
= 2ke^{2\xi}x^T(t)p_1x(t) + 2e^{2\xi}x^T(t)p_1\left(-Ax(t) + Wf_1(y(t - \tau(t)))\right)
\]
\[
\frac{dV_2(t)}{dt} = 2ke^{2\xi}y^T(t)p_2y(t) + 2e^{2\xi}y^T(t)p_2dy(t)
\]
\[
= 2ke^{2\xi}y^T(t)p_2y(t) + 2e^{2\xi}y^T(t)p_2\left(-By(t) + Vf_2(x(t - \sigma(t)))\right)
\]
\[
\frac{dV_3(t)}{dt} = \eta^T(t)\Xi\eta(t) + \xi^T(t)\Xi\xi(t) < 0
\]

Then, by applying S-procedure, \(\frac{dV(t)}{dt} < 0\), if there exist positive diagonal matrices \(U_1 = \text{diag}(u_{11}, u_{12}, \ldots, u_{1m})\) and \(U_2 = \text{diag}(u_{21}, u_{22}, \ldots, u_{2n})\), such that

\[
\frac{dV(t)}{dt} = -2e^{2\xi}\sum_{i=1}^{m}u_{i1}f_{ij}(y_j(t))\left[f_{ij}(y_j(t)) - l_{ij}y_j(t)\right] - 2e^{2\xi}\sum_{i=1}^{n}u_{i2}f_{ij}(x_i(t))\left[f_{ij}(x_i(t)) - l_{ij}x_i(t)\right]
\]

where

\[
\eta(t) = \left[x^T(t), x^T(t - \tau(t)), f_1^T(x(t)), f_1^T(y(t - \tau(t)))\right]^T
\]
\[
\xi(t) = \left[y^T(t), y^T(t - \tau(t)), f_2^T(y(t)), f_2^T(x(t - \sigma(t)))\right]^T
\]

Since the matrices \(\Xi < 0\) and \(\Sigma < 0\) in (3) and (4), we have \(\frac{dV(t)}{dt} < 0\), it follows that \(V(t) \leq V(0)\).

\[
V(0) \leq \left[\lambda_M(p_1) + d_Ml_{1M} + \lambda_M(q_1)l_{2M} + \lambda_M(r_1)\right] \frac{1 - e^{-2k\tau}}{2k}
\]

\[
\sup_{[-\max(\sigma,\tau),0]}\left\|x(t)\right\| + \left\|y(t)\right\| \leq \gamma\left[\sup_{[-\max(\tau,0),0]}\left\|x(t)\right\| + \sup_{[-\max(\sigma,\tau),0]}\left\|y(t)\right\|\right]
\]

\[
l_{1M} = \max\{l_{11}, l_{12}\}, l_{2M} = \max\{l_{21}, l_{22}\}, d_M = \max(d_1, d_2, \ldots, d_n), \epsilon_M = \max(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)
\]

Also, we have

\[
V(t) \geq e^{2\xi}\left[\lambda_M(p_1) + d_Ml_{1M} + \lambda_M(q_1)l_{2M} + \lambda_M(r_1)\right]\left(\left\|x(t)\right\| + \left\|y(t)\right\|\right)^2
\]

\[
\geq e^{2\xi}\min\left(\lambda_M(p_1), \lambda_M(q_1), \lambda_M(r_1)\right)\left(\left\|x(t)\right\| + \left\|y(t)\right\|\right)^2
\]

Therefore, we have

\[
\left\|x(t)\right\| + \left\|y(t)\right\| \leq \eta\left[\sup_{[-\max(\tau,0),0]}\left\|x(t)\right\| + \sup_{[-\max(\sigma,\tau),0]}\left\|y(t)\right\|\right]e^{-2k\tau}
\]

It is clear that \(\eta \geq 1\), up to now, we can come to a conclusion that (2) is globally exponential stable, that is, (1) is globally exponential stable. This completes the proof.

### 4. Example

Consider the BAM neural networks (2) with the following parameters: 

\[
\begin{align*}
\begin{cases}
 f_{ij}(y_j(t)) = \max(y(t) - l_{ij}y_j(t), 0), \quad &j = 1, 2, \ldots, m \\
 f_{ij}(x_i(t)) = \max(x(t - \sigma(t)) - l_{ij}x_i(t), 0), \quad &i = 1, 2, \ldots, n
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\frac{dx_i(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^{n} w_{ij} f_{x_j}(y_j(t-1)), \quad i = 1, 2, 3 \\
\frac{dy_j(t)}{dt} &= -b_j y_j(t) + \sum_{i=1}^{m} v_{ij} f_{y_i}(x_i(t-0.5)), \quad j = 1, 2, 3
\end{align*}
\]

where

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 0.05 & 0.10 & 0.15 \\ 0.25 & 0.05 & 0.15 \\ 0.05 & 0.15 & 0.05 \end{bmatrix}, \\
B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad V = \begin{bmatrix} 0.75 & 0 & 0.15 \\ 0.75 & 0.50 & 0.95 \\ 0.95 & 0.75 & 0.95 \end{bmatrix}
\]

\[
f_{x_i}(x_j) = \frac{1}{2}(|x_j + 1| - |x_j - 1|), \\
f_{y_i}(y_j) = \frac{1}{2}(|y_j + 1| - |y_j - 1|).
\]

So, we have

\[
L_1 = L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

The result in [10] is \( k = 0.35 \). By applying Theorem 1 in our paper, the obtained upper bound of an exponential convergence rate \( k = 0.5719 \), which shows that our criterion is less conservative than one in [10].

5. Conclusions

Based on Lyapunov stability theory and LMI technique, a new criterion is derived as LMI to ensure the exponential stability of BAM neural networks with time-varying delays in this letter. The previously used bounding technique has shown to be unnecessary in deriving the delay-dependent stability result. What is more, this approach is computational effective since it can be easily verified and solved by using of the LMI control toolbox in Matlab. The effectiveness of the proposed criterion is demonstrated in numerical example.

References


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