Abstract—In this paper, we propose a two-dimensional (2-D) angles of arrival (AOAs) estimation method based on a joint diagonalization of two spatio-temporal (ST) correlation matrices. The mathematical manipulations proposed in this paper take the structure of the array that enable estimating 2-D AOAs simultaneously without 2-D searching or pairing. The performance comparison shows that the proposed method is better than ST-DOA matrix method.

Index Terms—2-D angles of arrival, joint diagonalization, pair matching, spatio-temporal.

1. Introduction

The problem of two dimensional (2-D) angles of arrival (AOAs) has attracted a lot of attentions, especially in fields such as radar, sonar, communications, and seismology. Many 2-D AOAs estimation methods have been proposed recently\cite{1}-\cite{10}. The popular high-resolution techniques used to distinguish multiple closely spaced sources are MUSIC-type and ML\cite{1}-\cite{4} methods. These methods, however, are based on computational demanding multidimensional searching for spectral peaks in a 2-D domain and are thus not amenable to real-time implementations. Though another class of 2-D AOAs estimation based on L-shape arrays\cite{3},\cite{5},\cite{6} can release the computation burden and obtain good AOAs estimates, either complex pair matching procedure is required or angles estimation of source signals with common one dimensional (1-D) angles may be failed.

In some applications, the sources are stationary with different spectral contents\cite{11},\cite{12}. L. Jin proposed a spatio-temporal DOA matrix (ST-DOA) algorithm which takes advantages of the a priori in time domain\cite{7},\cite{8}. It can estimate the DOA with common 1-D angles; however, the signals in some curved surface cannot be resolved.

In this paper, we propose a 2-D AOAs estimation method based on a joint diagonalization of two spatio-temporal covariance matrices (JD-ST-DOA matrix algorithm). It is shown that performing a joint diagonalization of a combined set of these matrices provides an improved estimation of the 2-D AOAs over the aforementioned techniques in two aspects. First, signals with common 1-D angles or in any curved surface can be resolved. Second, robustness is increased at low signal to noise ratios (SNRs). Comparison of the proposed technique with the ST-DOA matrix method\cite{7},\cite{8} is presented.

2. Assumptions and Data Model

Consider an arbitrary array consisting of $M$-element as shown in Fig. 1 and the first three sensors are chosen as guiding sensors. The displacement vectors joining the two guiding sensor pairs are along $X$ and $Y$ axes, respectively, and their corresponding magnitudes are $d_x$ and $d_y$. $(\alpha_k, \beta_k)$ $(0 \leq \alpha_k, \beta_k < \pi, \quad k = 1,2,\cdots,D)$ denotes the AOAs of the $k$th signal $s_k(t)$. It is assumed that the source signal vector $s(t)$ is either H1, a deterministic ergodic sequence, or H2, a stationary multivariate process\cite{11},\cite{12}.

H1: $\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} s(t+\tau)s^H(t) = \text{diag}[\rho_1(\tau), \rho_2(\tau), \cdots, \rho_D(\tau)]$ \hfill (1)

H2: $E[s(t+\tau)s^H(t)] = \text{diag}[\rho_1(\tau), \rho_2(\tau), \cdots, \rho_D(\tau)]$ \hfill (2)

where superscript $H$ denotes the complex conjugate transpose.
of a vector, and $\text{diag}[\cdot]$ is the diagonal matrix formed with the elements of its vector valued argument. Assumptions H1 and H2 mean that the component processes $s_k(t)$ ($k=1,2,\cdots,D$) are mutually uncorrelated, and $\rho_k(\tau)$ denotes the autocovariance of $s_k(t)$, where superscript $H$ denotes the Hermitian conjugate of a vector.

Suppose $K$ snapshots are obtained, the baseband signals of the $r$th snapshot of the array output measured by the array can be expressed as

$$x_r(t) = \sum_{k=1}^{D} s_k(t) \exp\left[ j \frac{2\pi}{\lambda} \left[ d_{rk} \cos(\alpha_k) + d_{ik} \cos(\beta_k) \right] \right] + n_r(t)$$

where $t = T, 2T, \cdots, KT$, $T$ is the sampling interval.

Equation (3) can be written in matrix form:

$$x(t) = As(t) + n(t)$$

where

$$x(t) = [x_1(t), x_2(t), \cdots, x_M(t)]^T$$

$$n(t) = [n_1(t), n_2(t), \cdots, n_M(t)]^T$$

$$s(t) = [s_1(t), s_2(t), \cdots, s_M(t)]^T$$

$$A = [a_1, a_2, \cdots, a_k, \cdots, a_M]^T$$

$$a_k = [a_{ik}, a_{ik}, a_{ik}, a_{ik}]^T$$

$$a_{ik} = \exp\left[ j 2\pi d_j \frac{\sin(\beta_k)}{\lambda} \right], i = 2,3$$

where the matrix $A$ is unknown and is not rank deficient by assumption. The additive noise $n(t)$ is modeled as a stationary, temporally white, zero mean complex random process with variance $\sigma^2$, independence of the source signals. For simplicity, we also require $n(t)$ to be spatially white:

$$E[n(t+\tau)n^H(\tau)] = \sigma^2\delta(\tau)I_M$$

where $\delta(\tau)$ is the Kronecker delta, and $I_M$ denotes the $M \times M$ identity matrix.

3. JD-ST-DOA Matrix Algorithm

3.1 Forming Two ST-DOA Matrices

Under the above assumptions, we have the following cross-correlation of the array outputs:

$$R_{x,n}(\tau) = E[x(t+\tau)x^H(t)]$$

$$= \sum_{k=1}^{D} \left[ R_{x,n}(\tau)a_k^H \right] a_k^T, \quad (l=1,2,3, \quad i=1,2,\cdots,M)$$

$$\tau \neq 0, \quad \tau = -NT, -2T, \cdots, -NT$$

(9)

Based on (8) and $\tau \neq 0$, the correlated noise can be removed. Let $y_i(\tau)$ $(l=1,2,3)$ and $b(\tau)$, respectively, be:

$$y_i(\tau) = [R_{x,n}(\tau), R_{x,n}(\tau), \cdots, R_{x,n}(\tau)]^T$$

(10)

$$b(\tau) = [R_{x,n}(\tau)a_1, \cdots, R_{x,n}(\tau)a_i, \cdots, R_{x,n}(\tau)a_M]^T$$

(11)

Rewrite $y_i(\tau)$ into matrix form:

$$y_i(\tau) = Ab(\tau) \quad (l = 3)$$

$$y_i(\tau) = A\Phi_1 b(\tau) \quad (l = 1)$$

$$y_i(\tau) = A\Phi_2 b(\tau) \quad (l = 2)$$

(12)-(14)

where $\Phi_1$ and $\Phi_2$ are $D \times D$ matrices,

$$\Phi_1 = \text{diag}\left[ e^{\frac{j 2\pi d_j \cos(\alpha_1)}{\lambda}}, \cdots, e^{\frac{j 2\pi d_j \cos(\alpha_D)}{\lambda}} \right]$$

$$\Phi_2 = \text{diag}\left[ e^{\frac{j 2\pi d_j \cos(\beta_1)}{\lambda}}, \cdots, e^{\frac{j 2\pi d_j \cos(\beta_D)}{\lambda}} \right]$$

(15)

(16)

By collecting the “pseudo snapshots” at $2N$ lags $\tau = -NT, -2T, \cdots, -NT$, the “pseudo snapshots” data matrices are formed as follows:

$$X_l = [y_1(-NT), \cdots, y_1(-T), y_1(T), \cdots, y_1(NT)]$$

(17)

equations (12)-(14) can be rewritten into

$$X_1 = AB$$

(18)

$$X_1 = A\Phi_1 B$$

(19)

$$X_2 = A\Phi_2 B$$

(20)

where

$$B = [b_1(-NT), \cdots, b_1(-T), b_1(T), \cdots, b_1(NT)]$$

Define ST-DOA matrix as

$$R_1 = X_l[X_l]^T$$

(21)

$$R_2 = X_l[X_l]^T$$

(22)

where $[X_l]^T$ denotes pseudoinverse, i.e.,

$$[X_l]^T = X_l[X_lX_l]^T$$

(23)

Based on the principle of DOA matrix algorithm[7]-[10], by eigendecomposition, we have

$$R_1 A_1 = A_1 \Phi_1$$

(24)

$$R_2 A_2 = A_2 \Phi_2$$

(25)

Then the $(\alpha_k, \beta_k)$ $(k=1,2,\cdots,D)$ can be obtained by using the first three elements of $A_k$ according to (6) and (7) or using (15) and (16) after alignment of parameters $\phi_k(k=1,2,\cdots,D)$ and $\phi_h(h=1,2,\cdots,D)$ according to (26)

$$P = (A_2^H A_2)^{-1} A_2^H A_1$$

(26)

wherever there is a unity at entry of $P (k,h)$.

The above procedure means that an estimate can be obtained if and only if $\Phi_1$ or $\Phi_2$ have unequal entries. However, there is a degeneracy in the eigenvectors $A_k$ when two sources have the same $\alpha$ and a degeneracy in the
eigenvectors $A_2$ when two sources have the same $\beta$, thereby precluding the ability to determine $A_1$ or $A_2$ or both if only $R_1$ or $R_2$ is used. Of course, as $R_1$ and $R_2$ share the same set of eigenvectors $A$ originally, we will devise, in the next section, a JD based procedure to determine a common $A$.

### 3.2 JD Procedure

The first step of our JD procedure consisting of obtaining a whitening matrix $W$, i.e., a $M \times D$ matrix verifying:

$$WQW^H = WAA^HW^H = I$$  \hspace{1cm} (27)

where

$$Q_x = \frac{1}{2N}Xx^H = \frac{1}{2N} \sum_{n=1}^{N} y(nT_y)y(nT_y)^H$$  \hspace{1cm} (28)

and we assume that

$$\frac{1}{2N} \sum_{n=1}^{N} b(nT_y)b(nT_y)^H = \frac{1}{2N} \mathbf{BB}^H = I$$

and $b(\tau)$ has unit variance so that the dynamic range of $b(\tau)$ is accounted for by the magnitude of the corresponding column of $A$, which does not affect the estimation of the 2-D AOAs.

Equation (27) shows that if $W$ is a whitening matrix, then $WA$ is a $D \times D$ unitary matrix. It follows that for any whitening matrix $M \times D$, there exists a unitary matrix $U$ such that $WA = U$. As a consequence, matrix $A$ can be factored as

$$A = W^HU$$  \hspace{1cm} (29)

This whitening procedure reduces the determination of the $M \times D$ mixture matrix $A$ to that of a unitary $D \times D$ matrix $U$. The whitened process still obeys a linear model

$$Z_i = WX_i, \quad i = 1, 2, 3$$  \hspace{1cm} (30)

Define the following cross-correlation matrix between $Z_3$ and $Z_i$ ($i = 1, 2$):

$$G_i = \frac{1}{N}Z_iZ_i^H = \frac{1}{N}[WX_iX_i^HW^H]$$

$$= \frac{1}{N}[W(\Phi_1)BB^H(\Phi_2)^HW^H] = U\Phi_iU^H$$  \hspace{1cm} (31)

The second step of our JD procedure is to determine unitary factor $U$, which is obtained by performing a joint diagonalization\(^{[11]}\) of the combined set of $\tilde{G} = \{G_1, G_2\}$. The essential uniqueness of joint diagonalization is guaranteed by the \textit{Theorem} given in [11].

**Theorem 1.** Sufficiency condition: For $\Phi_1$ and $\Phi_2$, if $(\alpha, \beta) \in [0, \pi) \times [0, \pi)$, then

$$\exists g, g = 1, 2, \quad \forall 1 \leq p \neq q \leq D \quad \phi_{pq} \neq \phi_{gg}

Proof. If $(\alpha, \beta) \in [0, \pi) \times [0, \pi)$, for the AOAs of the $p$th source $(\alpha_p, \beta_p)$ and $q$th source $(\alpha_q, \beta_q)$, $p \neq q$, there exists three cases:

1. If $\beta_p \neq \beta_q$, $\alpha_p = \alpha_q$, then $\phi_{pq} \neq \phi_{qq}$;
2. If $\beta_p = \beta_q$, $\alpha_p \neq \alpha_q$, then $\phi_{pq} \neq \phi_{qq}$;
3. If $\beta_p \neq \beta_q$, $\alpha_p \neq \alpha_q$, then $\phi_{pq} \neq \phi_{qq}$.

Theorem 1 means that there exists at least one matrix $\Phi_g$ ($g = 1, 2$) satisfies $\phi_{pq} \neq \phi_{qq}$. Then matrix $A$ can be obtained by (29).

The third step of our JD procedure is to determine $(\alpha, \beta)$ of the $i$th source. Based on (24) and (25), as $a_k$ ($k = 1, 2, \cdots, D$) is the eigenvector of both $R_1$ and $R_2$, we can get

$$\phi_{k} = \cos^{-1}[\text{arg}(a_k^H R_i a_k)] / 2 \pi d_i$$  \hspace{1cm} (34)

$$\beta = \cos^{-1}[\text{arg}(a_k^H R_i a_k)] / 2 \pi d_i$$  \hspace{1cm} (35)

no pair matching operation needs to be done.

### 3.3 Implementation of the JD-ST-DOA Matrix Algorithm

Based on the previous sections, the JD-ST-DOA matrix algorithm is defined by the following implementation:

1. Estimate the sample cross-correlation of the array outputs according to (9).
2. Form the new pseudo-observation vectors $X_i$ ($i = 1, 2, 3$).
3. Estimate the sample covariance $Q_L$ from the $M \times T$ pseudo-snapshots $X_i$. Let $\lambda_1, \lambda_2, \cdots, \lambda_D$ denote the $D$ large eigenvalues of $Q_L$, and $h_1, h_2, \cdots, h_p$ be the corresponding eigenvectors. The whitening matrix $W$ is formed by

$$W = \{\lambda_1^{1/2}h_1, \lambda_2^{1/2}h_2, \cdots, \lambda_p^{1/2}h_p\}$$

4. Form the cross-correlation matrix $G_i$ ($i = 1, 2$) according to (31).
5. A unitary matrix $U$ is then obtained as joint diagonalizer of the set $G$.
6. The matrix $A$ is estimated as $A = W^U$, then the 2-D AOAs can be estimated according to (34) and (35).

### 3.4 Computation Complexity

In general, the total snapshots $K$ and pseudo snapshots $2N$ are much greater than the number of sensors, i.e., $K \gg M$, $2N \gg M$. Forming the $2N$ lags cross-correlation of the array
outputs requires on the order of \((2KNM-N^3M-NM)\). Forming the sample covariance matrix requires on the order of \(2M^2N\), eigendecomposition of a \(M \times M\) dimensional matrix requires on the order of \(O(M^3)\) \([13]\). Joint diagonalizer of the set \(\hat{G}\) (two \(D \times D\) matrices) requires on the order of \(2O(D^3)\) \([14]\). The main computation flops of JD-ST-DOA matrix algorithm lies in forming the \(2N\) lags cross-correlation of the array outputs.

4. Simulation Results

In our simulations, assume \(d_1 = d_2 = \lambda / 2\), \(M=4\). Simulation results are also compared with those of the ST-DOA matrix method \([7],[8]\).

Example 1: Assume three narrowband signals impinging on the array from directions \((50^\circ,50^\circ)\), \((50^\circ,70^\circ)\) and \((70^\circ,70^\circ)\). Note that in this case \(s_1\) and \(s_2\) have common \(\alpha\), \(s_2\) and \(s_3\) have common \(\beta\). The frequencies of these baseband signals, which are normalised by sample frequency, are 0.12, 0.14, 0.16, respectively. The snapshots \(K=150\) and the pseudo snapshots \(2N=100\) \((n \in [-50,-1] \cup [1,50])\). In the ST-DOA matrix algorithm, we use the third ST-DOA matrix since the first two ST-DOA matrices have degenerate eigenvalue spectra. The root mean square error (RMSE) of the \(k\)th source is defined as

\[
RMSE_k = \sqrt{E[(\hat{\alpha}_k - \alpha_k)^2 + (\hat{\beta}_k - \beta_k)^2]}.
\]

The performance of the estimators is obtained from 300 Monte-Carlo simulations, by calculating the RMSEs of the AOA estimates. Fig. 2 shows the RMSEs in degrees of the estimates of the three signals when SNR is varied from \(-10\) dB to \(10\) dB. We can see that the robustness is increased with our JD-ST-DOA method at low SNRs.

Example 2: In this example, we use three benchmark signals \(s_1, s_2, s_3\). They can be found in the file nband5.mat (the first three signals) provided by the ICALAB toolbox with \([15]\). Assume three signals impinging on the array from the directions \((50^\circ,50^\circ), (60^\circ,50^\circ), (69^\circ,69^\circ)\). In this case for ST-DOA matrix algorithm each ST-DOA matrix has a degenerate eigenvalue spectrum. \(K=300, 2N=200\) \((n \in [-100,-1] \cup [1,100])\), SNR=15 dB. To obtain a measure of statistical repeatability, we make 100 Monte-Carlo simulations. Fig. 3 shows that the ST-DOA matrix algorithm can only estimate one of the three signals because the other two signals have a degenerate eigenvalue spectrum, but the proposed method can estimate three signals successfully.

From above examples, it is clear that the proposed JD-ST-DOA matrix algorithm outperforms the ST-DOA matrix algorithm because the joint diagonalization criterion allows the structure information contained in each spatio-temporal correlation matrix to be jointly integrated in a single unitary matrix.

Fig. 3. Comparison of JD-ST-DOA with ST-DOA.

5. Conclusions

In this paper, we proposed a JD-ST-DOA matrix algorithm to handle the problem of 2-D AOA estimation. It is a direct approach with high resolution by using joint diagonalization technique without a search procedure. Moreover, the proposed algorithm can handle sources with common 1-D angles. Numerical examples illustrate that the proposed method is better than ST-DOA matrix method.

References


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