Stochastic Resonance in a Harmonic Oscillator Fluctuating Intrinsic Frequency by Asymmetric Dichotomous Noise

Shi-Qi Jiang, Bo Wu, and Tian-Xiang Gu

Abstract—The stochastic resonance phenomenon in a harmonic oscillator with fluctuating intrinsic frequency by asymmetric dichotomous noise is investigated in this paper. By using the random average method and Shapiro-Logino formula, the exact solution of the average output amplitude gain (OAG) is obtained. Numerical results show that OAG depends non-monotonically on the noise characteristics: intensity, correlation time and asymmetry. The maximum OAG can be achieved by tuning the noise asymmetry and or the noise correlation time.

Index Terms—Asymmetric dichotomous noise, harmonic oscillator, output amplitude gain, stochastic resonance.

1. Introduction

The mechanism of stochastic resonance (SR) was first proposed by Benzi et al. [1], [2] to describe the cooperative effect of the stochastic perturbation and periodic force leading to amplification of the peak of power spectrum. Early, people focused on non-linear system with periodic field and noise [3]. It seemed that all three ingredients of nonlinearity, periodic force, and random force are necessary for the onset of SR. Recently, it was demonstrated that SR can exist in linear system driven by periodic field and noise [4]-[14], too. The conventional SR, the SR in broad sense, and the bona fide SR denote non-monotonic dependence of the output characteristics, such as amplitude, moment, autocorrelation function, power spectrum, signal-to-noise ratio or dynamic parameters, on some noise characteristics and the external field frequency, respectively.

Harmonic oscillators subjected to a random force in various noise systems have long been a subject of study [15]. A quantitative investigation of harmonic oscillator was started in the remarkable article of Bourret et al. [16]. Fluctuation of external parameters of an oscillator is usually expressed by multiplicative noise. The internal dynamics of a force-free harmonic oscillator with random frequency was widely introduced as a model to understand the different phenomena in physics (on-off intermittency, dye lasers, polymers in random field), biology (population dynamics), economics (stock market prices), and so on. An investigation of the response of harmonic oscillator driven by an external periodic field with symmetric dichotomous noise demonstrated the existence of stochastic resonance [6], [9].

Dichotomous noise, i.e., random telegraph noise, is a two-state stochastic process. Random switching between discrete levels is known as dichotomous noise. Such switching may affect observable properties in many different ways and have different origins. Dichotomous noise has been observed in various systems such as in resonant tunneling diodes, metal-oxide-semiconductor structures, metallic nanoconstrictions, tunnel barriers, single atoms, and in single molecules. In solid-state circuits, dichotomous noise has most often been observed and is usually harmful to systems [17]. A unit with the random switching characteristic can potentially affect on the performances of the system. For example, digital switching noise impact on LC-tank voltage-controlled oscillators (VCOs) in lightly doped substrates [18].

Asymmetric dichotomous noise [16] in actual systems [19]-[23] can easier observed than symmetric. In asymmetric dichotomous model, when asymmetry becomes zero, the asymmetric dichotomous noise turns into symmetric dichotomous noise. There are a few papers that investigate stochastic resonance in an over-damped linear system driven by asymmetric dichotomous noise [7], [11]-[14]. But, there are few papers that investigate a harmonic oscillator driven by asymmetric dichotomous noise with fluctuating intrinsic frequency.

2. Model and Its OAG

We consider a forced linear oscillator with random frequency [8], [9] by the following model:

\[
\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + [\omega^2 + \zeta(t)]x = A\sin(\Omega t)
\]  

(1)

where \( x \) is the output of the oscillator, \( \gamma \) is damping coefficient, \( \omega^2 \) is intrinsic frequency, \( \Omega \) is frequency of external field, and \( \zeta(t) \) is a random intrinsic frequency.

Equation (1) is a motion equation of a harmonic oscillator with intrinsic frequency fluctuated by asymmetric dichotomous noise and driven by a periodic field. Various applications of this harmonic oscillator were discussed previously in [8], [9].

In our consideration, \( \zeta(t) \) is a two-valued random telegraph process with zero mean, i.e., asymmetric
dichotomous noise, it can be characterized\(^{[16]}\) by
\[
\xi(t) \in \{ \Delta_1, -\Delta_2 \} \\
\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \sigma \exp[-\lambda |t-t'|] \quad (2a)
\]
where \( \Delta_1 \) and \( -\Delta_2 \) (\( \Delta_1 > 0, \quad \Delta_2 > 0 \)) are two values of asymmetric dichotomous noise, \( \sigma = (\Delta_1,\Delta_2) \) and \( \lambda \) are the intensity and the correlate rate of asymmetric dichotomous noise, respectively, and \( \langle . \rangle \) denotes the statistical averaging.

Defining \( \Delta = \Delta_1 - \Delta_2 \) which denotes the asymmetry of the noise, from (2) one can prove
\[
\xi^2(t) = \sigma + \Delta \xi(t). \quad (3)
\]

Equation (1) can be rewritten as a set of first-order differential equations:
\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -2\gamma y - \omega^2 x - \xi x + A \sin(\Omega t)
\end{align*} \quad (4)
\]

After averaging, we take the following form:
\[
\begin{align*}
\frac{d}{dt} \langle x \rangle &= \langle y \rangle \\
\frac{d}{dt} \langle y \rangle &= -2\gamma \langle y \rangle - \omega^2 \langle x \rangle - \langle \xi x \rangle + A \sin(\Omega t)
\end{align*} \quad (5)
\]

Equation (5) contains a new correlator \( \langle \xi x \rangle \). By using the well-known Shapiro-Loginov formula\(^{[24]}\) for exponentially correlated noise in (2), we have
\[
\frac{d}{dt} \langle \xi x \rangle = \left( \frac{\xi}{dt} \right) \langle x \rangle - \lambda \langle \xi x \rangle = \langle \xi y \rangle - \lambda \langle \xi x \rangle. \quad (6)
\]

Similarly, for \( \langle \xi y \rangle \) in (6),
\[
\frac{d}{dt} \langle \xi y \rangle = \left( \frac{\xi}{dt} \right) \langle y \rangle - \lambda \langle \xi y \rangle = -(2\gamma + \lambda) \langle \xi y \rangle - \omega^2 \langle \xi x \rangle - \langle \xi^2 x \rangle. \quad (7)
\]

According to (3), we can find the higher-order correlation function \( \langle \xi^2 x \rangle \) as
\[
\langle \xi^2 x \rangle = \sigma \langle x \rangle + \Delta \langle \xi x \rangle. \quad (8)
\]

Substituting (8) into (7) yields
\[
\frac{d}{dt} \langle \xi y \rangle = -(2\gamma + \lambda) \langle \xi y \rangle - (\omega^2 + \Delta) \langle \xi x \rangle - \sigma \langle x \rangle. \quad (9)
\]

We thus obtain a system of (5), (6), and (9) for four variables \( \langle x \rangle, \quad \langle y \rangle, \quad \langle \xi x \rangle \) and \( \langle \xi y \rangle \). From these equations, one can easily find the fourth-order differential equation for \( \langle x \rangle \):
\[
\frac{d^4}{dt^4} \langle x \rangle + m_1 \frac{d^3}{dt^3} \langle x \rangle + m_2 \frac{d^2}{dt^2} \langle x \rangle + m_3 \frac{d}{dt} \langle x \rangle + m_4 \langle x \rangle = A m_5 \sin(\Omega t) + A m_6 \cos(\Omega t) \quad (10)
\]

where
\[
m_1 = 2(2\gamma + \lambda) \\
m_2 = (2\gamma + \lambda)^2 + 2\omega^2 + 2\gamma \lambda + \Delta \\
m_3 = (2\gamma + \lambda)(\omega^2 + 2\gamma \lambda) + 2\gamma(\omega^2 + \Delta) + \omega^2 \lambda \\
m_4 = (2\gamma + \lambda)\omega^2 \lambda + \omega^2(\omega^2 + \Delta) - \sigma \\
m_5 = \lambda(2\gamma + \lambda) + \omega^2 + \Delta - \Omega^2 \\
m_6 = 2\Omega(\gamma + \lambda).
\]

In the absence of a driving force and zero initial conditions, the mean solution \( \langle x \rangle \) should relax to zero.

According to the Routh-Hurwitz theorem, equation (10) is stable if the following relations between coefficients \( m_i \) \((i = 1, \ldots, 4)\) hold:
\[
m_1 > 0, \quad m_2 > 0, \quad m_3 > 0, \quad m_4 > 0 \\
m_5 < m_1 m_2, \quad m_1 m_3 m_4 - m_2 m_3 - m_5 > 0. \quad (11)
\]

The solution of (10) can be proved in the form:
\[
\langle x \rangle = \langle x \rangle_0 + \langle x \rangle_a \quad (12)
\]

where \( \langle x \rangle_0 \) is the output signal induced by an external field \( A \sin(\Omega t) \) and \( \langle x \rangle_0 \) is defined by internal dynamics. In the case that the system is stable, i.e., \( t \to \infty \), we have \( \langle x \rangle_0 \to 0 \) and \( \langle x \rangle_a \) is in the form:
\[
\langle x \rangle_a = a \sin(\Omega t + \phi). \quad (13)
\]

Then, we find
\[
|a/A| = \left| \frac{m_1^2 + m_2^2}{(\Omega^2 - \Omega^2 m_1^2 + m_3^2)^2 + (2\Omega m_1 - \Omega^3 m_1 m_3)^2} \right|^{1/2} \quad (14)
\]

and
\[
\phi = \arctan \left( \frac{(\Omega^2 - \Omega^2 m_2 - m_3 m_4 - (\Omega m_3 - \Omega^3 m_3 m_4) m_5)}{(\Omega^2 - \Omega^2 m_2 + m_3)m_6 + (\Omega m_3 - \Omega^3 m_3)m_4} \right). \quad (15)
\]

Note that \( |a/A| \) in (14) is the output amplitude gain (OAG) which is ratio of the output amplitude to the input amplitude. The OAG in (14) reaches a maximum when
\[
\sigma = \Omega^4 - \Omega^2[(2\gamma + \lambda)^2 + 2\omega^2 + 2\gamma \lambda + \Delta] \\
+ \omega^2(2\gamma + \lambda) + \omega^2(\omega^2 + \Delta). \quad (16)
\]

In the case of symmetry dichotomous noise, i.e., \( \Delta = 0 \), (14) becomes
\[
|a/A| = \left( \frac{S_1^2 + S_2^2}{S_1^2 + S_2^2} \right)^{1/2} \quad (17)
\]

where
\[
S_1 = \lambda(2\gamma + \lambda) + \omega^2 - \Omega^2 \\
S_2 = 4\Omega^2(\gamma + \lambda).
\]
\[
S_3 = \Omega^4 - \Omega^2[(2\gamma + \lambda)^2 + 2\omega^2 + 2\gamma \lambda] + (2\gamma + \lambda)\omega^2\lambda + \omega^4 - \sigma
\]
\[
S_4 = 4(2\gamma + \lambda)^2[\Omega(\gamma \lambda + \omega^2) - \Omega^3]
\]
and the dependence of a stationary signal amplitude on the correlation rate shows typical SR non-monotonic behavior for \( \Omega < \omega \). However, the heights of the maximal are non-monotonic functions of \( \Omega \).

In the case of white Gaussian noise, i.e., \( \Delta = 0 \), \( \sigma \to \infty \), \( \lambda \to \infty \) and \( \sigma/\lambda = D \), (14) takes the form:
\[
|a/\Delta| = \frac{1}{(\omega^2 - \Omega^2)^2 + 4\Omega^2}\gamma^2 .
\]
Equation (18) indicates that the noise hardly affect on the output amplitude gain for white Gaussian noise.

3. Discussion

According to (14), we discuss influences of noise, signal, and system parameters on the OAG.

Fig. 1 shows the relation between the OAG and the intensity \( \sigma \) of the asymmetry dichotomous noise specified by \( \gamma = 0.1 \), \( \omega = 1 \), \( \lambda = 1 \), and \( \Omega = 0.5 \). The curve exhibits a peak for different \( \Delta \) and the OAG is a non-monotonic function of \( \sigma \). This indicates that the conventional SR phenomenon occurs in the oscillator with asymmetry dichotomous noise. From Fig. 1, it can be observed that as increasing the asymmetry of asymmetric dichotomous noise, the peak moves to up-right. This observation implies that the asymmetry affects the characteristic of the conventional SR and causes the conventional SR to be stronger. From [8] and [9], we know that the conventional SR occurs with symmetric dichotomous noise too, but this is a special case of our result with \( \Delta = 0 \).

It must be emphasized that the influences of the asymmetry of the noise on OAG are non-monotonic, as shown in Fig. 2 for \( \omega = 1 \), \( \Omega = 0.5 \), \( \sigma = 1 \), and \( \lambda = 0.1 \). This means stochastic resonance in a general sense exists. From the curves of OAG, we can observe that the OAG has a maximum value along with the asymmetry increasing. So, the maximum OAG can be achieved by tuning the asymmetry of noise. Furthermore, OAG is enhanced as the damping reduces. This result was not involved in [8], [9].

Fig. 2. The dependence of OAG on the noise asymmetry.

For \( \gamma = 0.1 \), \( \omega = 1 \), \( \sigma = 0.8 \), and \( \Delta = 0.2 \), Fig. 3 shows the curves of the OAG versus the noise correlation time \( \tau = \lambda^{-1} \) with variations of the frequency of external field. As the frequency \( \Omega \) increases, the OAG rises with the increase of the noise correlation time in about \( \tau < 1 \), while the OAG decreases in about \( \tau > 1 \). That is, the frequency \( \Omega \) of external field enhances the OAG for small correlation time \( (\tau < 1) \), but reduces the OAG for large correlation time \( (\tau > 1) \). Therefore, the maximal OAG can be achieved by adjusting the correlation time.

Fig. 4 is curves of OAG as the external field frequency. The dashed line is without noise and the solid line is with noise \( \sigma = 0.5 \), \( \lambda = 1 \), and \( \Delta = 0.1 \). The resonant frequency (dynamic resonance) is slightly reduced when the intrinsic frequency of a harmonic oscillator is randomized by asymmetric dichotomous noise. The OAG of the system with noise is equal to that without noise at a certain frequency point (intersection between dashed and solid line). This is an interesting observation which might reveal some new understanding of stochastic resonance in systems.

Fig. 3. The dependence of OAG on the noise correlation time.

Fig. 4. The dependence of OAG on the frequency of the field for \( \gamma = 0.1 \), \( \omega = 1 \).
4. Conclusions

We have studied SR in a harmonic oscillator with asymmetric dichotomous noise. By applying Shapiro-Loginov formula, the expression of OAG is derived for fluctuating intrinsic frequency. The analysis presented shows that the OAG not only depend non-monotonically on the intensity and the correlation time of asymmetric dichotomous noise, but also a non-monotonic function of noise asymmetry.

In general, any system can not be without noise, people always think that influence of noise on performances of system is destructive. But by this paper, we show that strengthening noise in a certain range is beneficial to OAG in a harmonic oscillator with fluctuating intrinsic frequency driven by asymmetry dichotomous noise. It means that we can apply noise to enhance output performance of system.

References

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