Asymptotic Properties of a Dynamic Neural System with Asymmetric Connection Weights

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Abstract In this paper, based on new Lyapunov function, the asymptotic properties of the dynamic neural system with asymmetric connection weights are investigated. Since the dynamic neural system with asymmetric connection weights is more general than that with symmetric ones, the new results are significant in both theory and applications. Specially the new result can cover the asymptotic stability results of linear systems as special cases.

Key words asymmetric connection weights; global exponential stability; neural networks

The importance of global asymptotic stability (GAS) for neural networks, as well as the neural approach for solving optimization has attracted considerable attention, and considerable effort has been put into the investigation of their analysis and synthesis in recent years. Many condition for GAS of the network equilibrium point were given among previous study.

In this paper, we are concerned with the following dynamic neural system
\[
\frac{du}{dt} = -u + P_W(Wu + \alpha q)
\]
(1)
where \(q \in \mathbb{R}^l\), \(\alpha\) is a positive constant, 
\(W = E - \alpha M\), \(M\) is an \(l \times l\) matrix,
\(\Omega = \{u \in \mathbb{R}^l | d_i \leq u_i \leq h_i, i \in I\}\), \(I \subset L, L = \{1, 2, \cdots, l\}\), and 
\(R^l \to \Omega\) is a projection operator which is defined by
\[
P_W(u) = \left[ \begin{array}{c} P_{W_1}(u_1) \\ P_{W_2}(u_2) \\ \vdots \\ P_{W_l}(u_l) \end{array} \right]
\]
and for \(i \in L - I\), \(P_{W_i}(u_i) = u_i\); for \(i \in I\)
\[
P_{W_i}(u_i) = \begin{cases} d_i & u_i < d_i \\ u_i & d_i \leq u_i \leq h_i \\ h_i & u_i > h_i \end{cases}
\]

This dynamic system has two important properties. One is that its equilibrium point solves a linear variational inequality thus, from the viewpoint of continuous methods, it can solve linear and convex quadratic programming problems and bimatrix equilibrium points, and the analysis of piecewise linear resistive circuits\([1~3]\). Another property is that this dynamic system is easily implemented by using a circuit with a single layer of neurons, where \(W\) is called a weight matrix, and, thus, is very amenable for parallel implementation. Hence, Eq.(1) can be viewed as a useful neural model. So, the stability of related neural networks has been investigated in Refs.\([5~11]\), and, the global asymptotic stability of the neural system (1) with a symmetric connection weight matrix \(W\) has been studied in Ref.\([12]\). In this paper, by using the property of the gap function and projection technique, we explore the global asymmetric connection weights\([1,13,14]\). In particular, our new results cover the asymptotic stability results of linear systems when \(\Omega = R^l\).

1 Preliminaries

First, we make variable for the system (1), let
\[
y = Wu + \alpha q
\]
(2)
then
\[
\frac{dy}{dt} = W \frac{du}{dt} = W [-u + P_W(y)] = -Wu + WP_W(y) = -y + WP_W(y) + \alpha q
\]
(3)

Notice, we can rearrange the order of \(y_1, y_2, \cdots, y_l\), make the index be included in \(I\) fore the \(r\), consider \(I = \{1, 2, \cdots, r\}\), then
\[
L - I = \{r + 1, r + 2, \cdots, l\}
\]
then let
\[
y = \begin{pmatrix} y_{(1)} \\ y_{(2)} \end{pmatrix}, \quad P_W(y) = \begin{pmatrix} L(y_{(1)}) \\ y_{(2)} \end{pmatrix}
\]
\[
W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \quad q = \begin{pmatrix} q_{(1)} \\ q_{(2)} \end{pmatrix}
\]
So the system (3) can be decomposed as

\[
\begin{align*}
\frac{dy_1(t)}{dt} &= -y_1(t) + W_{11}L(y_1(t)) + W_{12}y_2(t) + \alpha q_{11} \\
\frac{dy_2(t)}{dt} &= -y_2(t) + W_{21}L(y_1(t)) + W_{22}y_2(t) + \alpha q_{12}
\end{align*}
\]  

(4)

Then

\[
\begin{align*}
\frac{dy_1(t)}{dt} &= -y_1(t) + W_{11}L(y_1(t)) + W_{12}y_2(t) + \alpha q_{11} \\
\frac{dy_2(t)}{dt} &= -y_2(t) + W_{21}L(y_1(t)) + W_{22}y_2(t) + \alpha q_{12}
\end{align*}
\]

\[y_1(t) = (y_1, y_2, \ldots, y_i)^T, \quad y_2(t) = (y_{r+1}, y_{r+2}, \ldots, y_T)^T \]

\[L(y_1(t)) = (P_{11}(y_1), P_{12}(y_2), \ldots, P_{22}(y_T))^T \]

\[q_{11} = (q_1, q_2, \ldots, q_T)^T, \quad q_{21} = (q_1, q_2, \ldots, q_T)^T \]

Notice \( W = E - \alpha M_2 \) if we make

\[M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \]

the relation between \( W \) and \( M_2 \) is as follows

\[
\begin{align*}
W_{11} &= E - \alpha M_{11} \quad W_{12} = -\alpha M_{12} \\
W_{21} &= -\alpha M_{21} \quad W_{22} = E - \alpha M_{22}
\end{align*}
\]

When \( i \in I \), from the definition of \( P_{ij}(u_i) \), it is easy to know output response function \( p_{ij}(y_i) \) (\( i \in I \)) satisfy

1) \( P_{ij}(y_i): R \rightarrow R \) is continuous function and have upper right derivative;

2) \( P_{ij}(y_i) \in R \), \( P_{ij}(y_i) \) is bounded function;

3) \( 0 \leq D^r P_{ij}(y_i) \leq 1 \).

**Theorem 1** If \( M_{22} \) is reversible, then the system (4) have equilibrium point.

**Proof** If the system (4) have equilibrium point \( y^* = (y_1^*, y_2^*, \ldots, y_T^*)^T \), then we have

\[
\begin{align*}
-y_1^{*1} + W_{11}L(y_1^{*1}) + W_{12}y_2^{*2} + \alpha q_{11} &= 0 \\
y_2^{*1} + W_{21}L(y_1^{*1}) + W_{22}y_2^{*2} + \alpha q_{12} &= 0
\end{align*}
\]

(5)

The following equation can be got

\[
\begin{align*}
y_1^{*1} &= [W_{11} + \frac{1}{\alpha} W_{12}M_{22}^{-1}W_{21}] L(y_1^{*1}) + \alpha q_{11} + W_{12}M_{22}^{-1}q_{21} \\
y_2^{*2} &= \frac{1}{\alpha} M_{22}^{-1}W_{21} L(y_1^{*1}) + M_{22}^{-1}q_{22}
\end{align*}
\]

then

\[
\begin{align*}
y_1^{*1} &= [E - \alpha (M_{11} + M_{12}M_{22}^{-1}M_{21})] L(y_1^{*1}) + \\
&\quad \alpha (q_{11} - M_{12}M_{22}^{-1}q_{21}) \\
y_2^{*2} &= -M_{22}^{-1}M_{21} L(y_1^{*1}) + M_{22}^{-1}q_{22}
\end{align*}
\]

(6)

Then

\[
\begin{align*}
y_1(t) &= [E - \alpha (M_{11} + M_{12}M_{22}^{-1}M_{21})] L(y_1^{*1}) + \\
&\quad \alpha (q_{11} - M_{12}M_{22}^{-1}q_{21}) \\
y_2(t) &= -M_{22}^{-1}M_{21} L(y_1^{*1}) + M_{22}^{-1}q_{22}
\end{align*}
\]

(7)

Therefore, the necessary and sufficient condition for system (4) to have equilibrium point is that Eq.(7) has solution.

Do mapping \( F: R^r \rightarrow R^r \), then

\[F(y) = Af(y) + b \]

(8)

and

\[
A = \begin{bmatrix} E - \alpha (M_{11} + M_{12}M_{22}^{-1}M_{21}) & 0 \\ -M_{22}^{-1}M_{21} & 0 \end{bmatrix}
\]

\[f(y) = \begin{bmatrix} L(y_1(t)) \\ 0 \end{bmatrix} \]

\[b = \begin{bmatrix} \alpha (q_{11} - M_{12}M_{22}^{-1}q_{21}) \\ M_{22}^{-1}q_{22} \end{bmatrix} \]

Define a convex set as

\[\Omega = \{x | x \in R^r \quad \text{and} \quad \|x - b\| \leq \|AM^*\|\} \]

then

\[
\|F(y) - b\| = \|Af(y)\| \leq \|A\| \|f(y)\| \leq \|A\| \|M^*\|
\]

from Eq.(8), we can know

\[
\|F(y) - b\| = \|Af(y)\| \leq \|A\| \|f(y)\| \leq \|A\| \|M^*\|
\]

where \( F \) is continuous mapping and \( F: \Omega \rightarrow \Omega, \) \( \Omega \)

is closed convex set from Brouwer fixed-point theorem, \( F \) has at least one fixed point \( y^* \), we have

\[y^* = F(y^*) = Af(y^*) + b\]

Namely the Eq.(2) have solutions, thereby we can know the system (4) have equilibrium point.

From Theorem 1, we can know that the system (4) has equilibrium points, let \( y^* = (y_1^*, y_2^*, \ldots, y_T^*)^T \) be equilibrium point for system (4), by means of the coordinate transformation:

\[x = y - y^*\]

then

\[
\begin{align*}
x(t) &= y(t) - y^* \\
x_1(t) &= y_1(t) - y_1^* \\
x_2(t) &= y_2(t) - y_2^*
\end{align*}
\]

then system(4) can be put into the equivalent
\[
\begin{align*}
\frac{dx_{t_0}}{dt} &= -x_{t_0} + W_{t_1}f(x_{t_0}) + W_{t_2}x_{t_2} \\
\frac{dx_{t_2}}{dt} &= -\alpha M_{22}x_{t_2} + W_{t_2}f(x_{t_0})
\end{align*}
\]

Then \( f(x_{t_0}) = L(x_{t_0}) + y_{t_0} - L(y_{t_0}) \), thus system (9) can be put into the equivalent system

\[
\begin{align*}
\frac{dx(t)}{dt} &= -x(t) + (E - \alpha M_{11})f(x(t)) - \alpha M_{12}x(t) \\
\frac{dx_{t_2}}{dt} &= -\alpha M_{22}x_{t_2} - \alpha M_{21}f(x(t))
\end{align*}
\]

(10)

2 Main Results

Theorem 2 If the system (10) satisfies

\[
\begin{align*}
\|E_r - \alpha M_{11}\| + \frac{\alpha}{2} (\|M_{12}\| + \|M_{21}\|) < 1 \\
\frac{1}{2} (\|M_{12}\| + \|M_{21}\|) < \lambda_{\min} \left( \frac{M_{22}^2 + M_{12}^2}{2} \right)
\end{align*}
\]

thus, the zero solution of the system (10) is globle exponential stability (GES), the trivial solution of the system (1) is GES.

Proof By the condition on \( \lambda_{\min}\left( \frac{M_{22}^2 + M_{12}^2}{2} \right) > 0 \), \( M_{22} \) is reversible, so the system (4) has equilibrium point, through coordinate varying, system (4) is turned into system (10). \( V(x) \) is positive definite. \( V(x) \) has infinitely great upper bounded and infinitesimal lower bounded. We use the Liapunov function for the system (10), then

\[
V(x) = x_{t_0}(t) x_{t_0}(t) + x_{t_2}(t) x_{t_2}(t)
\]

We see easily \( V(x) \) is positive definite and unbounded function. Along the trajectories of Eq.(4), the time derivative of \( V(x) \) is given by the following equation

\[
\frac{dV}{dt} = -2x_{t_0} f(x_{t_0}) + 2x_{t_0} (E_r - \alpha M_{11}) f(x_{t_0}) -
\]

\[
2\alpha x_{t_0}^T M_{12} x_{t_2} - 2\alpha x_{t_0}^T M_{22} x_{t_2}^2 + \frac{M_{22}^2 + M_{12}^2}{2} x_{t_2}^2 -
\]

\[
2\alpha x_{t_2}^T M_{12} f(x_{t_0}) = -2\|x_{t_0}\|^2 + 2\|E_r -
\]

\[
\alpha M_{11}\|x_{t_0}\|\|f(x_{t_0})\| + 2\alpha \|M_{12}\|\|x_{t_0}\|\|f(x_{t_0})\|
\]

\[
\|x_{t_2}\| - 2\lambda_{\min}\left( \frac{M_{22}^2 + M_{12}^2}{2} \right) x_{t_2}^2 +
\]

\[
2\alpha \|M_{22}\|\|x_{t_0}\|\|x_{t_2}\| \leq -2\|x_{t_0}\|^2 +
\]

\[
2\|E_r - \alpha M_{11}\|\|x_{t_0}\|^2 + \alpha \|M_{12}\|\|x_{t_0}\|^2 +
\]

\[
\|x_{t_2}\| - 2\alpha \lambda_{\min}\left( \frac{M_{22}^2 + M_{12}^2}{2} \right) x_{t_2}^2 +
\]

\[
\|x_{t_2}\|^2 - 2\alpha \lambda_{\min}\left( \frac{M_{22}^2 + M_{12}^2}{2} \right) x_{t_2}^2 +
\]

\[
\alpha \|M_{22}\|\|x_{t_0}\|^2 + \|x_{t_2}\|^2
\]

From the Eq.(11) we know \( \mu_1 > 0, \mu_2 > 0 \), such that

\[
\frac{dV}{dt} |_{(10)} \leq -2\mu_1 \|x_{t_0}\|^2 - 2\mu_2 \|x_{t_2}\|^2
\]

Take \( \mu = \min(\mu_1, \mu_2) > 0 \) again, we have

\[
\frac{dV}{dt} |_{(10)} \leq -2\mu \|x_{t_0}\|^2 + \|x_{t_2}\|^2 = -2\mu V(t)
\]

thus \( e^{2\mu t} V(x(t)) \leq e^{2\mu t} V(t_0) \).

\[
V(x(t)) = \|x(t)\|^2 \leq V(t_0) e^{-2\mu(t-t_0)} = \|x(t_0)\|^2 e^{-2\mu(t-t_0)}
\]

then

\[
\|x(t)\|^2 \leq \|x(t_0)\|^2 e^{-2\mu(t-t_0)}
\]

We can know the zero solution of the system is GES, thereby the solution of the system is GES. From the result, we can deduce that the equilibrium point of the system (10) is uniqueness, further, the equilibrium point of the system (1) is also uniqueness.

We conside weight formaly about the system (10)

\[
\begin{align*}
\frac{dx}{dt} &= -x + \sum_{i=1}^{r} (e_y - \alpha m_y) f(x) - \alpha \sum_{j=1}^{r} m_y x_j \\
\frac{dx}{dt} &= -\alpha \sum_{j=1}^{r} m_y x_j - \alpha \sum_{j=1}^{r} m_y f(x)
\end{align*}
\]

Then \( e_y = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \) \((i, j = 1, 2, \ldots, r)\), \( M = (m_y)_{r \times r} \).

Theorem 3 If the system (11) satisfies

\[
\sum_{j=1}^{r} |e_y - \alpha m_y| + \alpha \sum_{j=1}^{r} |m_y| < 1
\]

\[
\sum_{j=1}^{r} |m_y| < \lambda = \lambda_{\min}\left( \frac{M_{22}^2 + M_{12}^2}{2} \right)
\]

(13)
such that the zero solution of the system is GES, further, the trivial solution of the system is GES.

**Proof**  We use the Liapunov function for the system (12): $V = \sum |x_i|$. We see easily $V(x)$ is positive definite and unbounded function. Along the trajectories of system (12), the derivative of $V(x)$ is given by the following equation

$$D^+ V(x)|_{(12)} = \sum_{i=1}^{l} \frac{dx_i}{dt} \text{sgn}(x_i) =$$

$$= \sum_{i=1}^{l} \left( -x_i + \sum_{j=1}^{r} (e_{ij} - \alpha m_{ij}) f(x_i) - \alpha \sum_{j=1}^{r} m_{ij} x_j \right) \times$$

$$\left( \text{sgn}(x_i) + \sum_{j=1}^{r} \left( -\alpha \sum_{j=1}^{r} m_{ij} x_j - \alpha \sum_{j=1}^{r} m_{ij} f(x_j) \right) \right) \times$$

$$\left( \text{sgn}(x_i) + \sum_{j=1}^{r} \left( -\sum_{j=1}^{r} x_j f(x_j) - \sum_{j=1}^{r} (e_{ij} - \alpha m_{ij}) \right) \right) \times$$

$$f(x_i) \text{sgn}(x_i) + \alpha \sum_{j=1}^{r} \left( \sum_{j=1}^{r} m_{ij} x_j \right) \text{sgn}(x_i) -$$

$$\alpha (\text{sgn}(x_{i_1}), \text{sgn}(x_{i_2}), \ldots, \text{sgn}(x_i)) M_2 x_i +$$

$$\alpha \sum_{j=1}^{r} \sum_{j=1}^{r} |m_{ij}||x_i|$$

Let

$$\text{sgn}(x_{i_1}) = (\text{sgn}(x_{i_1}), \text{sgn}(x_{i_2}), \ldots, \text{sgn}(x_i))^T$$

$$\text{sgn}(x_{i_2})^T M_2 x_{i_2} = \left[ \text{sgn}(x_{i_2})^T \right] \left( \frac{M_2 + M_2^T}{2} \right) x_{i_2} \geq$$

$$\lambda_{\text{min}} \left( \frac{M_2 + M_2^T}{2} \right) (\text{sgn}(x_{i_2}))^T x_{i_2} =$$

$$\lambda_{\text{min}} \left( \frac{M_2 + M_2^T}{2} \right) \sum_{i=1}^{l} |x_i|$$

Let

$$\lambda = \lambda_{\text{min}} \left( \frac{M_2 + M_2^T}{2} \right)$$

$$D^+ V(x)|_{(12)} \leq -\sum_{i=1}^{l} |x_i| + \sum_{j=1}^{r} |e_{ij} - \alpha m_{ij}| |x_i| +$$

$$\alpha \sum_{j=1}^{r} \sum_{j=1}^{r} |m_{ij}||x_i| - \alpha \lambda \sum_{i=1}^{l} |x_i| + \alpha \sum_{j=1}^{r} \sum_{j=1}^{r} |m_{ij}| |x_i| -$$

$$|x_i| = \sum_{i=1}^{l} \left( 1 - \sum_{j=1}^{r} |e_{ij} - \alpha m_{ij}| - \alpha \sum_{j=1}^{r} |m_{ij}| \right) \times$$

$$|x_i| - \alpha \sum_{j=1}^{r} \left( \sum_{j=1}^{r} m_{ij} \right) |x_i|$$

Let

$$\mu_1 = \min_{\alpha \geq 0} \left( 1 - \sum_{j=1}^{r} |e_{ij} - \alpha m_{ij}| - \alpha \sum_{j=1}^{r} |m_{ij}| \right)$$

$$\mu_2 = \min_{\alpha \geq 0} \left( 1 - \sum_{j=1}^{r} |m_{ij}| \right)$$

From Eq. (13), we know $\mu_1 > 0$, $\mu_2 > 0$, let $\mu = \min(\mu_1, \alpha \mu_2)$, we have

$$D^+ V(x)|_{(12)} \leq -\mu \sum_{i=1}^{l} |x_i| - \alpha \mu_2 \sum_{i=1}^{l} |x_i| \leq -\mu \sum_{i=1}^{l} |x_i| = -\mu V$$

Thus

$$D^+ (e^{\mu t} V) = \mu e^{\mu t} V + e^{\mu t} D^+ V \leq 0$$

Obviously

$$e^{\mu t} V(x(t)) \leq e^{\mu t} V(x(t_0))$$

$$V(x(t)) = \sum_{i=1}^{l} |x_i(t)| \leq \sum_{i=1}^{l} |x_i(t_0)| e^{\mu(t-t_0)}$$

So the zero solution of the system (12) is GES, further, the trivial solution of the system (1) is GES.

## 3 Conclusions

In this paper, we have studied the global exponential stability of a dynamic neural system with asymmetric weights using two new Liapunov function. The obtained stability result of system (1) naturally generalizes the stability result of linear systems, in contrast to the existing results by Forti and Tesi, which cannot cover linear dynamic systems. As asymmetric weight cases are more common than symmetric ones, the obtained results are very useful from both theoretical and applicational points of view.

References


(Continued on page 86)