Product Approximation of Grade and Precision*

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Abstract The normal graded approximation and variable precision approximation are defined in approximate space. The relationship between graded approximation and variable precision approximation is studied, and an important formula of conversion between them is achieved. The product approximation of grade and precision is defined and its basic properties are studied.

Key words rough sets; approximation operators; operators of approximations; graded rough sets; variable precision rough sets

The classical rough set model, proposed by Pawlark based on equivalence relation, has some limitations[1]. In practice there are many complicated relations. So it is necessary to improve the classical model. Different models have been presented and studied, such as the model on a normal relation[2,3], the graded rough set model applied in general approximate space[4], and the variable precision rough set model[1,4,5]. The latter two models have extended classical models from cardinal and precision. The grade and precision of a model are confidence levels, so they are important for applications.

In this paper, a graded rough set model is first defined in approximate space. The precision of variable rough set model is generalized to [0, 1], this has been studied in Ref.[6]. The relationship and conversion between the two models are studied. Furthermore, the product approximation of grade and precision is defined and its properties are studied.

1 Preliminaries

Assume that \(U\) is the universe, \(A\) is a set, \(R\) is an equivalence relation on \(U, U/R\) is the quotient set, \([x]_R\) is the \(R\) equivalence class containing \(x\), \((U, R)\) is the approximate space, then

\[
\overline{R}A = \bigcup \{[x]_R / [x]_R \cap A \neq \phi\}
\]
\[
\underline{R}A = \bigcup \{[x]_R / [x]_R \subseteq A\}
\]

are called \(R\) upper and lower approximations of \(A\), \(\overline{R}\) and \(\underline{R}\) are called the upper, lower approximation operators.

Let \(k\) be 0 or natural number, real number \(\beta \in [0, 0.5]\), then

\[
\overline{ar}_k A = \{x \in U / \overline{R}(x) \cap A > k\}
\]
\[
\underline{ar}_k A = \{x \in U / \underline{R}(x) \cap A \leq k\}
\]

are called grade \(k\) \(R\) upper and lower approximations of \(A\), here

\[
\overline{R}_\beta A = \bigcup \{[x]_R / c([x]_R) \cap A > 1 - \beta\}
\]
\[
\underline{R}_\beta A = \bigcup \{[x]_R / c([x]_R) \cap A \leq \beta\}
\]

are called precision \(\beta\) \(R\) upper and lower approximations of \(A\).

2 Relationship and Conversion of Graded and Variable Precision Approximations

Definition 1 Assume that \((U, R)\) is the approximate space, \(k\) is 0 or natural number, real number \(\beta \in [0, 1]\), then

\[
\overline{R}_A = \bigcup \{[x]_R / [x]_R \cap A > k\}
\]
\[
\underline{R}_A = \bigcup \{[x]_R / [x]_R \cup A \leq k\}
\]

are called grade \(k\) \(R\) upper and lower approximations of \(A\); and

\[
c([x]_R, A) = 1 - \bigcup \{[x]_R / c([x]_R) \cap A \}
\]
\[
\overline{R}_\beta A = \bigcup \{[x]_R / c([x]_R) \cap A > 1 - \beta\}
\]
\[
\underline{R}_\beta A = \bigcup \{[x]_R / c([x]_R) \cap A \leq \beta\}
\]

are called precision \(\beta\) \(R\) upper and lower approximations of \(A\). If \(\overline{R}_A = \overline{R}_A\), then \(A\) is definable set by grade \(k\), otherwise \(A\) is rough set by grade \(k\). If \(\overline{R}_A = \overline{R}_A\), then \(A\) is definable set by
precision \( \beta \), otherwise \( A \) is rough set by precision \( \beta \).

**Theorem 1**  The graded rough set model has the following properties:

1. \( \overline{R}_\beta \phi = \phi \quad \overline{R}_\beta U = U \)
2. \( \overline{R}_\beta \phi = \bigcup \{ [x]_{k} | [x]_{k} \leq k \} \)
3. \( \overline{R}_\beta U = \bigcup \{ [x]_{k} | [x]_{k} > k \} \)
4. \( A \subseteq B \Rightarrow \overline{R}_\beta A \subseteq \overline{R}_\beta B \quad \overline{R}_\beta A \subseteq \overline{R}_\beta B \)
5. \( R_\beta (A \cup B) \supseteq R_\beta A \cup R_\beta B \)
6. \( R_\beta (A \cap B) \supseteq R_\beta A \cap R_\beta B \)
7. \( R_\beta (\sim A) = \sim R_\beta A \quad R_\beta (\sim A) = \sim R_\beta A \)

The variable precision rough set model has the similar properties:

1. \( \beta \neq 1 \), then
2. \( \overline{R}_\beta \phi = \overline{R}_\beta \phi = \phi \quad \overline{R}_\beta U = \overline{R}_\beta U = U \)
3. \( \forall A \subseteq U \Rightarrow \overline{R}_\beta A \subseteq \overline{R}_\beta B \quad \overline{R}_\beta A \subseteq \overline{R}_\beta B \)
4. \( R_\beta (A \cup B) \supseteq R_\beta A \cup R_\beta B \)
5. \( R_\beta (A \cap B) \supseteq R_\beta A \cap R_\beta B \)
6. \( R_\beta (\sim A) = \sim R_\beta A \quad R_\beta (\sim A) = \sim R_\beta A \)
7. \( \beta \geq \alpha \Rightarrow \overline{R}_\beta A \supseteq \overline{R}_\alpha A \quad \overline{R}_\beta A \supseteq \overline{R}_\alpha A \)

**Definition 2**  \( \forall k \) and \( \forall [x]_{k} \in U/R \),

\[
\beta([x]_{k}, k) = k/|[x]_{k}|
\]

is called grade \( k \) error degree of \( [x]_{k} \). \( \forall \beta \in [0, 1] \) and \( \forall [x]_{k} \in U/R \),

\[
k([x]_{k}, \beta) = \beta |[x]_{k}|
\]

is called precision \( \beta \) grade of \( [x]_{k} \).

In terms of transformation and the definition of \( c([x]_{k}, A) \), it is easy to prove

\[
c([x]_{k}, A) < 1 - \beta \Leftrightarrow [x]_{k} \cap A \supseteq k([x]_{k}, \beta)
\]

\[
c([x]_{k}, A) \leq \beta \Leftrightarrow [x]_{k} \cap A \supseteq k([x]_{k}, \beta)
\]

**Theorem 2**  \( \overline{R}_\alpha A = \overline{R}_\beta ([x]_{k} / [x]_{k} \cap A) \)

\[
\overline{R}_\alpha A = \bigcup \{ [x]_{k} | [x]_{k} \cap A \supseteq k([x]_{k}, \beta) \}
\]

In the two models, parameters \( k \) and \( \beta \) are strict rules whose actions are on all the equivalence classes. From this theorem we can see, the grade error degree and the precision grade are similar to the precision and grade respectively except for number field and range. In appearance the two models can be converted each other based on grade error degree and precision grade. In this paper, from the conversion relationship we use the grade form. Of course it can be described by precision form too.

### 3 The Product Approximations of Grade and Precision

**Definition 3**  \( \overline{R}_\beta A = \overline{R}_\beta (A_{\beta}) \)

\[
R_\beta A = R_\beta (A_{\beta})
\]

are called product \( R \) upper and lower approximations of grade \( k \) and precision \( \beta \).

\[
\overline{R}_{\beta \alpha} A = \overline{R}_{\beta \alpha} (A_{\beta \alpha})
\]

\[
R_{\beta \alpha} A = R_{\beta \alpha} (A_{\beta \alpha})
\]

are called product \( R \) upper and lower approximations of precision \( \beta \) and grade \( k \).

**Theorem 3**  \( \overline{R}_{\beta \alpha} A = \bigcup \{ [x]_{k} | [x]_{k} \cap A \supseteq k([x]_{k}, \beta) \}
\]

\[
\bigcup \{ [x]_{k} | [x]_{k} \subseteq R_{\beta \alpha} A \}
\]

Especially, if \( \beta = 1 \), then \( \overline{R}_{\beta \alpha} A = \phi \) and \( R_{\beta \alpha} A = U \).

2) If \( \beta = 1 \), then \( \overline{R}_{\beta \alpha} A = \phi \) and \( R_{\beta \alpha} A = U \). If \( \beta \neq 1 \), then \( \overline{R}_{\beta \alpha} A = \overline{R}_{\beta \alpha} A \) and \( R_{\beta \alpha} A = R_{\beta \alpha} A \).

**Proof**  1) Since \( R_{\beta \alpha} A = R_{\beta}(R_{\beta \alpha} A) \), from the definition of \( R_{\beta} \) and the properties of classification, \( [x]_{k} \cap R_{\beta \alpha} A > k \) if and only if \( [x]_{k} \subseteq R_{\beta \alpha} A \) and \( [x]_{k} \supseteq k \). So

\[
\overline{R}_{\beta \alpha} A = \bigcup \{ [x]_{k} | [x]_{k} \supseteq k([x]_{k}, \beta) \}
\]

Since \( R_{\beta \alpha} A = R_{\beta}(R_{\beta \alpha} A) \), from the definition \( R_{\beta} \)}
there is $|[x]_k| \leq k$. The case $|[x]_k| \leq k$ is obvious. If $|[x]_k| > k$, then
$$[x]_k \cap R_\beta A \not= [x]_k \setminus [x]_k \leq k$$
So $[x]_k \subseteq R_\beta A$. The contrary can be got easily by definition. Therefore,
$$R_\beta A = (\bigcup [x]_k / [x]_k \leq k)$$
Especially, since $\bar{R}_\beta A = \phi$ and $R_\beta A = U$, if $\beta = 1$ then $\bar{R}_\beta A = \phi$ and $R_\beta A = U$.

2) Since $\bar{R}_\beta A = \bar{R}_\beta (R_k A)$, from the definition of $\bar{R}_\beta$, there is $|[x]_k \cap R_\beta A| > \beta |[x]_k|$. If $\beta = 1$, then $[x]_k$ does not exist, so $R_\beta A = \phi$. If $\beta \neq 1$, then $|[x]_k \cap R_\beta A| > \beta |[x]_k| \Rightarrow [x]_k \subseteq R_\beta A$, so $\bar{R}_\beta A = \bar{R}_\beta A$.

Since $R_\beta A = R_\beta (R_k A)$, from the definition of $R_\beta$, there is $|[x]_k \cap R_\beta A| \geq [x]_k \leq R_\beta A$, so $R_\beta A = R_\beta A$. If $\beta = 1$, then $[x]_k \subseteq U$, so $\bar{R}_\beta A = U$. If $\beta \neq 1$, then $|[x]_k \cap R_\beta A| \geq [x]_k \leq R_\beta A$, so $\bar{R}_\beta A = \bar{R}_\beta A$. We can see that $|[x]_k| \leq k$ is a special case. In fact, from definition, $\forall A \subseteq U$, if $|[x]_k| \leq k$, then $[x]_k \subseteq R_\beta A$ and $[x]_k \subseteq R_\beta A$. In other words, $|[x]_k| \leq k$ is a sufficient condition of $[x]_k \subseteq R_\beta A$; and $|[x]_k| > k$ is a necessary condition of $[x]_k \subseteq R_\beta A$. Just because of this, we have the essence of product approximations of grade and precision.

$\bar{R}_\beta A$ and $\bar{R}_\beta A$ are commonplace when $\beta = 1$, and when $\beta \neq 1$ they are the approximations of grade, whose properties are presented in other paper. Here we will only study the properties of $\bar{R}_\beta A$ and $\bar{R}_\beta A$.

**Theorem 4**

1) $\bar{R}_\beta \phi = \phi$ $\bar{R}_\beta U = U$ $\bar{R}_\beta \phi = U$ $\bar{R}_\beta U = \phi$

If $\beta = 1$, then
$$\bar{R}_\beta U = \bigcup [x]_k / [x]_k \leq k$$

2) If $\beta \in \{0.5, 1\}$, then $\bar{R}_\beta A \subseteq \bar{R}_\beta A$; and if $\beta \in \{0.5, 0.5\}$, then $\bar{R}_\beta A \subseteq \bar{R}_\beta A \cup \bigcup [x]_k / [x]_k > k$).

3) $A \subseteq B \Rightarrow \bar{R}_\beta A \subseteq \bar{R}_\beta B, \bar{R}_\beta A \subseteq \bar{R}_\beta B$.

4) $\bar{R}_\beta (A \cup B) \supseteq \bar{R}_\beta A \cup \bar{R}_\beta B$

5) $\bar{R}_\beta (\bar{R}_\beta A \cup B) \supseteq \bar{R}_\beta A \cup \bar{R}_\beta B$

6) $\bar{R}_\beta (\bar{R}_\beta A) = \bar{R}_\beta A$

7) $\bar{R}_\beta (\bar{R}_\beta A) = \bar{R}_\beta A$

**Proof**

1) $\bar{R}_\beta \phi = \phi$ since $\bar{R}_\beta \phi = \phi$ and $R_\beta U = U$ since $R_\beta U = U$. If $\beta = 1$, then $\bar{R}_\beta A = U$ and $\bar{R}_\beta A = \phi$. If $\beta \neq 1$, then $\bar{R}_\beta \phi = \phi$ and $R_\beta U = U$, so $\bar{R}_\beta \phi = \phi$. If $\beta = 0$, then $\bar{R}_\beta \phi = \phi$ and $R_\beta U = U$, so $\bar{R}_\beta \phi = \phi$.

2) If $\beta \in \{0.5, 1\}$ then $\bar{R}_\beta A \subseteq \bar{R}_\beta A$, so $\bar{R}_\beta A \subseteq \bar{R}_\beta A$; similarly, if $\beta \in \{0.5, 0.5\}$ then $\bar{R}_\beta A \subseteq \bar{R}_\beta A$. We have
$$\bar{R}_\beta (\bar{R}_\beta A) = \bigcup [x]_k / [x]_k > k, [x]_k \subseteq \bar{R}_\beta (\bar{R}_\beta A)$$

From the properties of classification and
$$\bar{R}_\beta A = \bigcup [x]_k / [x]_k \leq k$$

so $\bar{R}_\beta A = \bigcup [x]_k / [x]_k \leq [x]_k \subseteq \bar{R}_\beta A$.

Similarly,
$$\bar{R}_\beta (\bar{R}_\beta A) = \bigcup [x]_k / [x]_k \leq k$$

From the properties of classification and
$$\bar{R}_\beta A = \bigcup [x]_k / [x]_k \leq k$$

so $\bar{R}_\beta A = \bigcup [x]_k / [x]_k \leq k$. Therefore
$$\bar{R}_\beta (\bar{R}_\beta A) = \bar{R}_\beta A$$
If $\beta \leq 0.5$, then
$$R_{\beta, \rho} A \subseteq \overline{R}_{\beta, \rho} A \cup (\cup \{ [x]_k / [x]_k \leq k \})$$
If $0.5 \leq \beta < 1$, then
$$R_{\beta, \rho} A \subseteq \overline{R}_{\beta, \rho} A \cup (\cup \{ [x]_k / [x]_k \leq k \})$$
3) If $\beta = 1$, then
$$\overline{R}_{\beta, \rho} A = \phi$$
$$R_{\beta, \rho} A = U$$
$$R_{\beta, \rho} A = U$$
If $\beta \neq 1$, then
$$R_{\beta, \rho} A = R_{\beta, \rho} A$$
$$\overline{R}_{\beta, \rho} A \cup (\cup \{ [x]_k / [x]_k \leq k \}) \subseteq R_{\beta, \rho} A$$
4) If $\beta < 0.5$, then
$$\overline{R}_{\beta, \rho} A \cup (\cup \{ [x]_k / [x]_k \leq k \}) = R_{\beta, \rho} A$$

**Proof**

1) If $[x]_k \subseteq \overline{R}_{\beta, \rho} A$, from the essence of $\overline{R}_{\beta, \rho}$, there is
$$\overline{R}_{\beta, \rho} A \subseteq \overline{R}_{\beta, \rho} A.$$ On the contrary, if $[x]_k \subseteq \overline{R}_{\beta, \rho} A$ and
$$R_{\beta, \rho} A = \cup \{ [x]_k / [x]_k \geq k \},$$
then $[x]_k \geq k$,
$$[x]_k \geq [x]_k \cap A > \beta [x]_k$$
so
$$[x]_k \subseteq \overline{R}_{\beta, \rho} A.$$
2) The case of $\beta = 1$ is easy to prove. When $\beta \neq 1$,
$$\overline{R}_{\beta, \rho} A \subseteq (\cup \{ [x]_k / [x]_k \leq k \})$$
$$\cup (\cup \{ [x]_k / [x]_k \geq k \}, [x]_k \cap A > \beta [x]_k)$$
If $[x]_k \subseteq \overline{R}_{\beta, \rho} A$, then $[x]_k \leq k$ or $[x]_k \geq k$ and $[x]_k \subseteq R_{\beta, \rho} A$. So
$$R_{\beta, \rho} A \subseteq \overline{R}_{\beta, \rho} A \cup (\cup \{ [x]_k / [x]_k \leq k \})$$
When $0.5 \leq \beta < 1$, $\beta \geq (1 - \beta) [x]_k$.
$$[x]_k \leq \overline{R}_{\beta, \rho} A.$$
If $[x]_k \leq \overline{R}_{\beta, \rho} A$, i.e., $[x]_k \geq k$, $[x]_k \cap A > \beta [x]_k$,
then $[x]_k \cap \overline{R}_{\beta, \rho} A = \cup \{ [x]_k / [x]_k \leq k \}$.
$$[x]_k \cap \overline{R}_{\beta, \rho} A = \cup \{ [x]_k / [x]_k \leq k \}$$
i.e., $\overline{R}_{\beta, \rho} A \subseteq \overline{R}_{\beta, \rho} A$. Obviously,
$$\cup \{ [x]_k / [x]_k \leq k \} \subseteq R_{\beta, \rho} A$$
So
$$R_{\beta, \rho} A = \overline{R}_{\beta, \rho} A \cup (\cup \{ [x]_k / [x]_k \leq k \})$$
3) When $\beta = 1$, it is easy to prove $\overline{R}_{\beta, \rho} A = \phi$, $R_{\beta, \rho} A = U$ and $R_{\beta, \rho} A = U$.
$$R_{\beta, \rho} A = (\cup \{ [x]_k / [x]_k \leq k \})$$
$$\cup (\cup \{ [x]_k / [x]_k \geq k \}, [x]_k \cap A > (1 - \beta) [x]_k)$$
so
$$R_{\beta, \rho} A = (\cup \{ [x]_k / [x]_k \leq k \})$$
$$\cup (\cup \{ [x]_k / [x]_k \geq k \}, [x]_k \cap A > (1 - \beta) [x]_k)$$
When $[x]_k \subseteq R_{\beta, \rho} A$, if $[x]_k \leq k$, then there is $[x]_k \subseteq R_{\beta, \rho} A$; if $[x]_k > k$, then there is $[x]_k \subseteq R_{\beta, \rho} A$ since $\beta \neq 1$.
On the contrary, when $[x]_k \subseteq R_{\beta, \rho} A$, if $[x]_k \leq k$, then there is $[x]_k \subseteq R_{\beta, \rho} A$. If $[x]_k > k$, then there is $[x]_k \subseteq R_{\beta, \rho} A$ since $[x]_k \geq 1 = \beta \subseteq R_{\beta, \rho} A$.
4) If $\beta < 0.5$, then $[x]_k \subseteq R_{\beta, \rho} A$.
$$R_{\beta, \rho} A = \cup \{ [x]_k / [x]_k \leq k \}$$
if $[x]_k \leq \overline{R}_{\beta, \rho} A$, then $[x]_k \subseteq R_{\beta, \rho} A$.
Then,
$$\overline{R}_{\beta, \rho} A \cup (\cup \{ [x]_k / [x]_k \leq k \}) \subseteq R_{\beta, \rho} A$$
4) When $\beta < 0.5$, if $[x]_k \subseteq R_{\beta, \rho} A$, since
$$R_{\beta, \rho} A = (\cup \{ [x]_k / [x]_k \leq k \})$$
$$\cup (\cup \{ [x]_k / [x]_k \geq k \}, [x]_k \cap A \geq (1 - \beta) [x]_k)$$
then $[x]_k \leq k$ or $[x]_k \geq k$.
And
$$[x]_k \cap A \geq (1 - \beta) [x]_k$$
and
$$\overline{R}_{\beta, \rho} A = \cup \{ [x]_k / [x]_k \leq k \}, [x]_k \cap A \geq (1 - \beta) [x]_k$$
so
$$[x]_k \subseteq \overline{R}_{\beta, \rho} A$$
and for $3)$
$$\overline{R}_{\beta, \rho} A \cup (\cup \{ [x]_k / [x]_k \leq k \}) = R_{\beta, \rho} A$$
In essence, the product approximations of grade and precision are the results of the combinations of approximations operators of grade and variable precision. Therefore, similarly to the above studies, we can study other combinations, such as $R_{\beta, \rho}$, $R_{\beta, \rho}$, $R_{\beta, \rho}$, $R_{\beta, \rho}$, and combinations for more than twice.

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