The terminology and notion in this paper are similar to Ref. [1]. All graphs discussed here are finite and simple.

The diameter \( d(G) \) of a graph \( G \) is the maximal distance between pairs of vertices of \( G \). The connectivity of \( G \) is the minimum number of vertices to be removed in order to disconnect the graph. When a network is modeled as a graph, a vertex represents a node of processor (or a station) and an edge between two vertices is the link (or connection) between those two processors. In this context, diameter is a measurement for maximum transmission delay, and connectivity is a good parameter to study how much tolerant the network can be in the occasion of node failures. Sometimes, we are interested in looking at a collection of multipaths between a pair of two vertices rather than at a single shortest path between them. So the \( k \) collection of multipaths between a pair of two vertices is the link (or connection) between those two processors. In this context, diameter is a measurement for maximum transmission delay, and connectivity is a good parameter to study how much tolerant the network can be in the occasion of node failures. Sometimes, we are interested in looking at a collection of multipaths between a pair of two vertices rather than at a single shortest path between them. So parameters of the \( k \)-wide distance (or \( k \)-distance) and \( k \)-diameter are introduced. They are extension of the distance and the diameter. The \( k \)-diameters of some kinds of graphs (or networks) have been studied in Ref. [2]. In this paper we discuss the \( k \)-distance and \( k \)-diameter of circular graph. And the \( k \)-diameters of the connected circulant graphs with degree 3 are obtained.

Lemma 1 If \( G \) is a 3-regular 3-connected graph with \( 2n \) vertices, then \( d_3(G) \leq n \).

Let

\[
\begin{align*}
Z_n &= \{0, 1, \ldots, n-1\} \\
S &\subseteq Z_n - \{0\} \\
- S &= S \mod n
\end{align*}
\]

namely, there exist \( j_1, j_2, \ldots, j_r \) such that

\[
S = \{j_1, j_2, \ldots, j_r, n - j_1, n - j_2, \ldots, n - j_r\}
\]

where \( j_1, j_2, \ldots, j_r \) are called spanning elements.

Definition 1 The graph \( G \) with order \( n \) is called circulant graph if it satisfies:

1. \( V(G) = Z_n \); 
2. \( E(G) = \{(j) | j - i \in S\} \), where the operation takes module \( n \).

The graph \( G \) in definition 1 is denoted by \( C_n(j_1, j_2, \ldots, j_r) \), where \( j_1 < j_2 < \cdots < j_r \). Clearly, according to definition 1, the circulant graphs are all regular graphs. So we also call a circulant graph as the circulant graph with degree \( k \) if it is a \( k \)-regular graph. Thus \( C_n(i, n/2) \), \( n \) is even, is a circulant graph with order \( n \) and degree 3 whose spanning elements are \( i \) and \( n/2 \).

Let \( \gcd(x, y) \) be the maximum common divisor of \( x \) and \( y \). It has been proved that a circulant graph \( C_n(j_1, j_2, \ldots, j_r) \) is a connected graph iff
Lemma 2 Let $G$ be a circulant graph of order $n$, and $k$ be a positive integer. If $d_k(G)$ is existent, then $\exists u \in \{1, 2, \cdots, \lfloor n/2 \rfloor \}$ satisfies

$$d_k(G) = d_k(0, u)$$

where

$$\lfloor n/2 \rfloor = \begin{cases} n/2 & n \text{ is even} \\ n-1/2 & n \text{ is odd} \end{cases}$$

or

$$d_k(G) = \max_{x \in A} \{d_k(0, x)\}$$

where $A = \{1, 2, \cdots, \lfloor n/2 \rfloor \}$

**Theorem 1** Let circulant graph $G = C_n(i, n/2)$, where $n \geq 4$ and $n$ is even. If $\gcd(n, i) = 1$, then

1) $d_i(G) = \lfloor n/2 \rfloor \quad \text{if } n \equiv 2 \pmod{4}$

2) $d_i(G) = \lfloor n/2 \rfloor + 1 \quad \text{if } n \not\equiv 2 \pmod{4}$

3) $d_i(G) = \frac{n}{2}$

**Proof** $G$ is a connected circulant graph means $\gcd(n, 2, n/2) = 1$. It follows that $n/2$ is odd.

1) According to the structure of $G$ as shown in Fig.1, for $\forall x \in \{0, 2, \cdots, n/2 - 2\}$, we have

$$d(0, \frac{n}{2} + x) = d(0, x) + 1$$

Since $x \in V(C_{n/2})$, we have

$$d_i(G) = \max_{x \in V(C_{n/2})} d(0, \frac{n}{2} + x) = \lfloor n/4 \rfloor + 1 = \frac{n + 2}{4}$$

where $C_{n/2} = 2 \cdots (n - 2) \cdot 0$ is a $n/2$-cycle. $N/2$ is odd means $\lfloor n/4 \rfloor = (n - 2)/4$.

2) It is easily seen that $d_i(C_n(2, 3)) = 3$. Now we suppose that $n > 6$ and $G' = G - 0$. According to the structure of $G'$ as shown in Fig.2, we know $y = 2(n - 2)/4 = (n - 2)/2$ and $y' = n/2 + y$. $N/2$ is odd means $\lfloor (n - 2)/4 \rfloor = (n - 2)/4$.

Let

$$p_0 = 2 \cdots y$$

$$q_0 = (n - 2)(n - 4) \cdots (y + 2)$$

Since $n/2$ is odd, we have

$$|p_0| = |q_0| = \frac{n - 2}{4} - 1$$

Let

$$p^* = p_0 y^*$$

$$q^* = q_0 (y^* + 2) y^*$$

$$r^* = \frac{n}{2} + 2(\frac{n}{2} + 4) \cdots y^*$$

Clearly,

$$|p^*| = |r^*| = \frac{n - 2}{4}$$

$$|q^*| = |p^*| + 1$$

According to Fig.2, the paths $p^*$, $q^*$ and $r^*$ are the shortest $2\cdot y^*$ path (that is the path from 2 to $y^*$), $(n - 2)\cdot y^*$ path and $(n/2)\cdot y^*$ path, respectively in $G'$. So, in $G$, the path $0p^*$ is the shortest 0-$y^*$ path which contains the vertex 2, $0q^*$ is the shortest 0-$y^*$ path which contains $n - 2$ and $0r^*$ is the shortest 0-$y^*$ path which contains $n/2$.

Let $N(0)$ be the neighbour set of the vertex 0 in $G$,
are also two internally disjoint 0-y' paths which both contain the vertices of N(0), we have

$$\max \{|0p'|, |0r'|\} \leq \max \{|p|, |q|\}$$

Therefore,

$$d_s(0, y') = \max \{|0p'|, |0r'|\} = \frac{n-2}{4} + 1 = \frac{n+2}{4}$$

We choose $x \in \{1, 2, \cdots, n/2\}$.

**Case 1** $x = n/2$. Let $p$ and $q$ are two internally disjoint 0-(n/2) paths, where $p$ is the path 0(n/2) and $q$ is the path 02(n/2+2)/2. Therefore,

$$d_s(0, \frac{n}{2}) \leq \max \{|p|, |q|\} = 3 = d_s(0, y') \quad (1)$$

**Case 2** $x \in \{2, 4, \cdots, y\}$, where $y = (n/2)/2$.

**Case 2.1** $x = y$. Let $p$ and $q$ are two internally disjoint 0-x paths, where $p$ is the path 02(4-2)/2 and $q$ is the path 0(n/2)(x-4)-x. Therefore,

$$d_s(0, x) \leq \max \{|p|, |q|\} = |q| = |p| + 1$$

$$\frac{n+2}{4} = d_s(0, y') \quad (2)$$

**Case 2.2** $x < y$. Let $p$ and $q$ are two internally disjoint 0-x paths, where $p$ is the path 02-4 and $q$ is the path 0(n/2)(n/2+2)-4-x. Therefore,

$$d_s(0, x) \leq \max \{|p|, |q|\} = |q| = |p| + 2$$

$$\frac{x}{2} + 2 < \frac{y}{2} + 2 = \frac{n+2}{4} = d_s(0, y') \quad (3)$$

**Case 3** $x \in \{1, 3, \cdots, (n/2)-2\}$. Let $p$ and $q$ are two internally disjoint 0-x paths, where $p$ is the path 0(n/2)(n/2-4)-4-x and $q$ is the path 0(n/2)(n/2-4)-x. Therefore

$$d_s(0, x) \leq \max \{|p|, |q|\} = |q| = |p| + 1$$

$$\frac{n+2}{4} = d_s(0, y') \quad (4)$$

By Eqs.(1), (2), (3) and (4), we have $d_s(0, x) \leq d_s(0, y')$, for $x \in \{1, 2, \cdots, n/2\}$. So, by Lemma 2 we have

$$d_s(G) = d_s(0, y') = \frac{n+2}{4}$$

3) $p' = (n/2+2)(n/2+4)\cdots(n/2-4)(n/2-2)$ as shown in Fig.3 is the shortest 0-(n/2-2) path in $G' = G-{n-2,n/2}$. On the other hand $p'$ is the shortest 0-(n/2-2) path in $G$ that contains neither the vertex $n-2$ nor the vertex $n/2$.

$q'$ and $r'$ are also two internally disjoint 0-(n/2-2) paths, where $q' = (n-2)(n/2-2)$ and $r' = (n/2)(n/2-2)$. Furthermore, $p'$, $q'$ and $r'$ are internally disjoint. Cleanly,

$$|q'| = |r'| \leq |p'|$$

It follows that for arbitrary three internally disjoint 0-(n/2-2) paths of $G$, $p$, $q$, and $r$, we have

$$\max \{|p'|, |q'|, |r'|\} = |p'| = \max \{|p|, |q|, |r|\}$$

Thus

$$d_s(0, \frac{n}{2} - 2) = |p'| = \frac{n}{2}$$

that is

$$d_s(G) \geq \frac{n}{2}$$

According to Fig.1 it is easily seen that there exist three internally disjoint $x$-$y$ paths in $G$, where $x$, $y$, $x \neq y$, are any pair vertices of $G$. Thus, by Menger’s Theorem, $G$ is a 3-connected graph. From Lemma 1, $d_s(G) \leq n/2$. So $d_s(G) = n/2$.

![Graph $G'$](image)

**Theorem 2** Let $G = C_n(i, n/2)$ be a connected circulant graph, where $n \geq 6$ and $n$ is even. If $\gcd(n, i) \neq 1$, then

1) $d_s(G) = \frac{n+2}{2}$

2) $d_s(G) = \begin{cases} 3 & n = 6 \\ \frac{n+2}{4} & n > 6 \end{cases}$

3) $d_s(G) = \frac{n}{2}$

**Proof** For a connected circulant graph $C_n(i, n/2)$, Ref.[5] shows that if $\gcd(n, i) \neq 1$ then $C_n(i, n/2) \cong C_n(2, n/2)$. Thus, by Lemma 3, 1), 2) and 3) all are true.

From Theorem 1 and Theorem 2, we have following Theorem 3 which deals with the $k$-diameters.
of the connected circulant graphs of degree 3.

Theorem 3 Let circulant graph \( G = C_n(i, n/2) \), \( n \geq 4 \) and \( n \) is even. If \( G \) is a connected graph, then

1) \( d_1(G) = \left\lfloor \frac{n + 2}{4} \right\rfloor \)

\[ \begin{cases} 
\left\lfloor \frac{n + 2}{4} \right\rfloor + 1 & \text{gcd}(n,i) = 1 \\
\left\lfloor \frac{n + 2}{4} \right\rfloor & \text{gcd}(n,i) \neq 1 \quad n > 6
\end{cases} \]

2) \( d_2(G) = \left\lfloor \frac{n + 2}{4} \right\rfloor \text{gcd}(n,i) \neq 1 \quad n > 6 \)

3) \( d_3(G) = \frac{n}{2} \quad n = 6 \)

3 Conclusions

When \( \text{gcd}(n, i) = 1 \), the \( k \)-diameter of \( C_n(i, n/2) \) has been studied in Ref.[4]. In this paper, for \( \text{gcd}(n, i) \neq 1 \) the \( k \)-diameters of \( C_n(i, n/2) \) is discussed. However, it is a difficult to obtain a \( k \)-diameter of arbitrary a circulant graph. On the other hand, the Cayley graph is a graph that has been widely used as a group-theoretic model for symmetric interconnection. The circulant graph is a kind of Cayley graph. therefore, it is necessary to further research the diameter of circulant graph.

References


Brief Introduction to Author(s)

ZHANG Xian-di (张先迪) was born in Sichuan Province, China, in 1947. He is now a professor with University of Electronic Science and Technology of China. His research interests include graph theory and its applications.

LI Man-li (李曼荔) was born in Sichuan Province, China, in 1982. He is now a postgraduate with School of Applied Mathematics, University of Electronic Science and Technology of China. His research interests include graph theory and its applications.

References


Brief Introduction to Author(s)

ZHANG Xian-yong (张贤勇) was born in Sichuan province, China, 1978. He graduated from College of Mathematics and Software Science of Sichuan Normal University in 2004. Now he is a teacher of Section of Fundamental Studies, Sichuan Normal University. His research includes mathematical logic and artificial intelligence. He has published more than 10 papers.

MO Zhi-wen (莫智文) is a Professor with College of Mathematics and Software Science, Sichuan Normal University. His research interests include theory and applications of fuzzy automaton.