On Zitterbewegung of Electron

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Abstract Traditionally, all the discussions on zitterbewegung (zbw) of electron are based on relativistic quantum mechanics. In this article, from the viewpoint of quantum field theory and in terms of the creation and annihilation operators as well as the polarization vectors of spin-1 field, a more detailed description and some new perspectives for zbw are obtained. Especially, it is shown that zbw arises from a to-and-fro vacuum polarization that occurring in the neighborhood of electron; the zbw vectors form a vector triplet with total spin projections 0 and ±1 in the direction of the momentum of electron, respectively; the macroscopic velocity of the vacuum medium vanishes in all inertial systems.

Key words Dirac electron; zitterbewegung; spin

Since the notion of zitterbewegung (zbw) was introduced by Schrödinger, the subject of zbw of a free particle has been examined in many articles[1-11]. However, all these works is examined only at the relativistic quantum mechanical level, where the descriptions for zbw are based on the concept of wave function rather than that of field operator. In this article, we study the zbw in the framework of quantum field theory and obtain a more explicit and detailed description and some new understandings for zbw.

1 The Helicity Vectors of Spin-1 Field

In our formalism, the helicity vectors of spin-1 field play an important role. It is discussed in detail as follows. Let \( i = \sqrt{-1} \), the spin-projection operator of spin-1 field in the direction of the momentum \( \mathbf{p} \) is

\[
\mathbf{\tau} \cdot \mathbf{p} / |\mathbf{p}| = \mathbf{\tau}_i, \mathbf{\tau}_2, \mathbf{\tau}_3 \text{ for } i = 1,2,3
\]

\( \mathbf{\tau}_i \) are the spatial spin matrices of spin-1 field, where

\[
\begin{align*}
\mathbf{\tau}_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \\
\mathbf{\tau}_2 &= \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \\
\mathbf{\tau}_3 &= \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

Let \( \mathbf{\tau} \cdot \mathbf{p} \mathbf{\eta}_i = |\mathbf{p}| \lambda_i \mathbf{\eta}_i, \ i = 1,2,3 \), we have

\[
\begin{align*}
\mathbf{\eta}_1 &= [\eta_{11}, \eta_{12}, \eta_{13}]^T \quad \text{for } \lambda_1 = 1 \\
\mathbf{\eta}_2 &= [\eta_{21}, \eta_{22}, \eta_{23}]^T = \eta^*_1 \quad \text{for } \lambda_2 = -1 \\
\mathbf{\eta}_3 &= [\eta_{31}, \eta_{32}, \eta_{33}]^T \quad \text{for } \lambda_3 = 0
\end{align*}
\]

where

\[
\begin{align*}
\eta_{11} &= \frac{1}{\sqrt{2}} \left( \mathbf{p}_1 - \mathbf{p}_2 \right) \\
\eta_{12} &= \frac{1}{\sqrt{2}} \left( \mathbf{p}_1 + \mathbf{p}_2 \right) \\
\eta_{13} &= \frac{1}{\sqrt{2}} \left( \mathbf{p}_1 - \mathbf{p}_2 \right) \\
\eta_{23} &= \frac{1}{\sqrt{2}} \left( \mathbf{p}_1 + \mathbf{p}_2 \right) \\
\eta_{33} &= \frac{1}{\sqrt{2}} \left( \mathbf{p}_1 \right)
\end{align*}
\]

and \( \eta^*_i \) is the complex conjugate of \( \eta_i \). Obviously, \( \eta_i, i = 1,2,3 \) form a complete orthonormal basis. The three 3×1 matrices \( \mathbf{\eta}_1, \mathbf{\eta}_2, \mathbf{\eta}_3 \) correspond to the three 3D vectors:

\[
\mathbf{\eta}_i = (\eta_{i1}, \eta_{i2}, \eta_{i3}) \quad i = 1,2,3
\]

Especially, in the 3D spatial coordinate system \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) with \( \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{p} / |\mathbf{p}| \), Eq.(2) becomes

\[
\begin{align*}
\mathbf{\eta}_1 &= \frac{1}{\sqrt{2}} (\mathbf{e}_1 + i\mathbf{e}_2) \\
\mathbf{\eta}_2 &= \frac{1}{\sqrt{2}} (\mathbf{e}_1 - i\mathbf{e}_2) \\
\mathbf{\eta}_3 &= \mathbf{e}_3
\end{align*}
\]
momentum $p$ is perpendicular to $\eta_1$ and $\eta_2$ while parallel to $\eta_3$. Therefore, Eq.(2) corresponds to the general spinor representation of the vector basis $\{e_1, e_2, e_3\}$. For example, the right-hand and left-hand circular polarization vectors are denoted by $\eta_1$ and $\eta_2$, respectively, while $\eta_3$ is the longitudinal polarization vector.

2 Traditional and New Description for Zitterbewegung

Let $\psi(x)=\psi(x,t)$, $p\cdot x = p^\mu x_\mu = E t - p \cdot x$, the Gordon decomposition of the free electron current $j^\mu = \bar{\psi}(x)\gamma^\mu \psi(x)$ is

$$j^\mu = \frac{1}{2m} [\bar{\psi} \gamma^\mu \psi - (\bar{p} \gamma^\mu \psi) - \frac{i}{m} \gamma^\mu \bar{\psi}\sigma^\mu \psi]$$

(4)

where $\mu, \nu, \rho = 0,1,2,3$, $\bar{\psi}(x)$ is the adjoint of the electron field operator $\psi(x)$ (related to the Hermitian adjoint $\psi^*$ by $\bar{\psi} = \psi^*$), $\gamma^\mu$ is the usual Dirac matrices, $m$ is the mass, $S^\mu = i(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)/4$ is the 4D spin tensor[12]. Let $\epsilon^{ijk}$ denotes the totally anti-symmetric tensor with $\epsilon^{123} = 1$ $(i,j,k = 1,2,3)$, one can show that $\Sigma_i = (\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$ with $\Sigma_i = \epsilon^{ijk}\Sigma_{jk}/2$ is the usual spin matrices (as the generators of 3D spatial rotations). We call the generators $K = (S_0, S_1, S_2, S_3)$ of Lorentz boosts as spin-like matrices. Then 3D spatial components of $j^\mu$ are

$$j = \psi^* \mathbf{a} \psi = [\bar{\psi} \mathbf{p} \psi - (\bar{p} \psi)\psi]/2m + [\nabla \times (\bar{\psi} \Sigma \psi)]/m - [\bar{\psi}(\Sigma \mathbf{K})\psi]/m$$

(5)

where $\mathbf{p} = -i\mathbf{\nabla}$ is the 3-dimensional momentum operator. In term of the Pauli’s matrix vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, we have

$$\Sigma = \frac{1}{2} \begin{bmatrix} 0 & -i\sigma & 0 \\ i\sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K = \frac{1}{2} \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

(6)

We define $m = e\Sigma/m$ and $n = eK/m$ ($e$ is the unit charge) as the intrinsic magnetic- and electric-moment operators of electron, respectively. Clearly, the first term on the right-hand side of Eq.(5) is the so-called convection current, while the remaining two terms are the contributions coming from the magnetic moment $m$ and the electric moment $n$, respectively.

Firstly, at the relativistic quantum mechanical level, we regard $\psi(x)$ as the wave function rather than the field operator, so the Hermitian adjoint $\psi^*$ should be replaced with the complex adjoint $\psi^\dagger$. And therefore $\psi(x)$ can be written as the following general form:

$$\psi(x) = \sum_{\nu} \int_{\mathbb{R}^3} \frac{m}{\sqrt{\mathbf{E}}} (c(p,s)u(p,s)\exp(-i\mathbf{p} \cdot \mathbf{x}) + d^\dagger(p,s)v(p,s)\exp(i\mathbf{p} \cdot \mathbf{x}))$$

(7)

where $s = 1,2$, corresponding to the spin $\pm 1/2$, respectively. $E = p_0$ is the energy, $V$ is the volume and $c(p,s) = c(p_0, p, s)$, $c(-p,s) = c(p_0, -p, s)$, etc., using Eq.(7), Eq.(5) can be integrated

$$\iiint j d^3x = \iiint \psi^\dagger a\psi d^3x = v + z$$

(8)

where $z$ is the so-called zbw current, and

$$v = \sum_{\nu} \frac{p_0}{\mathbf{E}} [\mathbf{c}(p,s)\mathbf{c}\dagger + d(p,s)\mathbf{d}\dagger]$$

(9a)

$$z = 2\text{Re} \left[ \sum_{\nu, \nu'} \mathbf{c}(p,s)\mathbf{d}\dagger(-p,s')\exp(-i2\mathbf{E}t) \right]$$

(9b)

where $s, s' = 1,2$. Obviously, $v$ corresponds to the group velocity. One can easily show that

$$\frac{1}{m} \iiint \mathbf{\nabla} \times (\bar{\psi} \Sigma \psi) d^3x = 0$$

(10a)

$$\frac{1}{m} \iiint \frac{\partial}{\partial t}(\bar{\psi} \Sigma \mathbf{K} \psi) d^3x = z$$

(10b)

Therefore, in Eq.(5) the current related to the magnetic moment $m$ does not contribute to the zbw current $z$, it is the current that related to the electric moment $n$ does contribute to the zbw current $z$. Obviously, the integration $\int z d^3y$ gives the spatial magnitude of the zbw motion.

Then, let us study the zbw in the framework of quantum field theory. For this we take $\psi$ as field operator, and correspondingly make the replacement $d^\dagger \rightarrow d^*$, where $d^*$ is the complex adjoint of $d$, while $d^\dagger$ the Hermitian adjoint, and so on. That is, the probability magnitudes are replaced by the creation and annihilation operators of field. In term of the vectors $\eta_i = (\eta_1, \eta_2, \eta_3)$ $(i = 1,2,3)$ given by Eq.(2), Eq.(8) becomes

$$\iiint j d^3x = \iiint \psi^\dagger a\psi d^3x = v + z_1 + z_2$$

(11)

where
\[ v = \sum_{p} \frac{p \cdot c'(p, s)c(p, s) - d'(p, s)d(p, s)}{E} \]  
(12a)

\[ z_v = \sum_{p} \sqrt{\eta} [c'(p, s)d'(-p, s)] \exp(i2E) = c(-p, s)d(p, s) \exp(-i2E)] - c(-p, s)d(p, s) \exp(-i2E)) \]  
(12b)

\[ z_\perp = \sum_{p} \sqrt{\eta} [c'(p, s)d'(-p, s)] \exp(i2E) = c(-p, s)d(p, s) \exp(-i2E)) \]  
(12c)

where \( s, s' = 1, 2 \),  \( \lambda_1 = 1 \), and  \( \lambda_2 = -1 \).

Here \( z_v \) and \( z_\perp \) represent two kinds of zb\( \perp \) vectors. In contrast to Eqs.(8) and (9), Eq.(12) give a more explicit and detailed expression for the zb\( \perp \) vectors, by the creation and annihilation operators as well as the helicity vectors of spin-1 field.

From the viewpoint of quantum field theory we discuss Eq.(12) as follows.

1) Consider that \( \tilde{N} = c'(p, s)c(p, s) \) and \( \tilde{N}' = d'(p, s)d(p, s) \) represent the particle-number operators of negative and positive electron, respectively, then \( v \) represents the velocity of the relativistic inertia-center of the system.

2) The operators \( \tilde{O} = c'(p, s)d'(-p, s) \) and \( \tilde{O}' = c(-p, s)d'(p, s) \) for \( s, s' = 1, 2 \) represent the creation and annihilation operators of particle-antiparticle pairs, and they result in vacuum polarization or vacuum fluctuation. In other words, the zb\( \perp \) motions described by \( z_v \) and \( z_\perp \) are related to vacuum polarization (fluctuation) occurring in the neighborhood of electron.

3) In our case, as for the particle-antiparticle pair produced by vacuum polarization, we define its center-of-mass velocity as the macroscopic velocity of the Dirac-vacuum medium. Let \( \{0\} \) denotes the vacuum state with \( \{0\} \) as its Hermitian adjoint, in view of the fact that the momentum \( p \) is arbitrary and the total momentum of \( \tilde{O}\{0\} \) or \( \{0\}\tilde{O}' \) always vanishes, the macroscopic velocity of the vacuum medium vanishes in all inertial systems.

4) Because of \( \eta_\perp \parallel p \), the zb\( \perp \) vector \( z_v \) is parallel to the momentum \( p \). (\( z_v \) is a longitudinal zb\( \perp \) vector. One can see that the total spin of the states \( z_v \{0\} \) or \( \{0\}z_v \) is 0, therefore \( z_v \) play the role of a longitudinal-polarization vector field.

5) Because of \( p \cdot \eta_\perp = p \cdot \eta_\perp = 0 \), the zb\( \perp \) vector \( z_\perp \) is vertical to the momentum \( p \). One can see that

the total spin of the states \( z_\perp \{0\} \) or \( \{0\}z_\perp \) is ±1, then \( z_\perp \) play the role of a transverse-polarization vector field, which includes the right-hand (related to \( \eta_\perp \)) and left-hand (related to \( \eta_\perp \)) circular-polarization vector fields.

6) Judged by the integrals \( \int z_v d'dt \) and \( \int z_\perp d'dt \), the spatial magnitudes are \( m/2E^2 \) for the longitudinal-zb\( \perp \) motion and \( E/2 \) for the right- and left-hand circular-zb\( \perp \) motions. Furthermore, the zb\( \perp \) velocities are \( \pm m/E \) in the longitudinal direction and ±1 in the transverse direction (\( h = c = 1 \)). Moreover, in the \( \lim p \to 0 \), the zb\( \perp \) vectors \( z_v \) and \( z_\perp \) do not vanish, that is, the zb\( \perp \) motions are intrinsic ones that is independent of the macroscopic motion of electron.

### 3 The 4D Spin Tensor of Electron

We will show that the zb\( \perp \) motions offer intrinsic dynamic degrees of freedom (i.e. the 4D spin tensor) for electron. Since \( x' = t \) and \( \partial x' / \partial t = (1, a) \), we have

\[ e\psi^\dagger (\partial / \partial t) \psi = e\psi^\dagger \gamma^\mu \psi = J^\mu \]  
(13)

We define the 4D electromagnetic-moment tensor of electron as

\[ \tilde{M}^{\mu \nu} = e(x^\mu \partial / \partial t - x^\nu \partial / \partial t) / 2 = \]  
\[ e \left( x^\mu m \partial / \partial t - x^\nu m \partial / \partial t \right) \]  
(14)

of which components correspond to the electric and magnetic moment of electron, respectively. The definition Eq.(14) holds in both classic and quantum cases. It is easy to show that \( J^\mu \) and \( V^\nu \) are the relativistic mass and 4D-velocity operators of electron, respectively. Due to the zb\( \perp \), \( V^\nu \) is different from the usual 4D-velocity. We define

\[ \tilde{\mathcal{L}}^{\mu \nu} = mV^\mu \]  
(15)

as the 4D instantaneous momentum and the 4D instantaneous orbital angular momentum, respectively. Here all the differences between the instantaneous and the usual quantities result from the fact that the 4D-velocity operator is different from the usual 4D-velocity, i.e., from the zb\( \perp \) motions. Using \( \mathcal{G} = \psi^* \gamma^0 \) one can show that

\[ \psi^* \tilde{M}^{\mu \nu} \psi = \frac{e}{2m} \mathcal{G}(x^\mu \tilde{\mathcal{L}}^{\mu \nu} - x^\nu \tilde{\mathcal{L}}^{\mu \nu}) \psi = \frac{e}{2m} \mathcal{G} \tilde{\mathcal{L}}^{\mu \nu} \psi \]  
(16)

If \( \psi^* \) and \( \psi \) both refer the same state, by applying
the Dirac equation and $\gamma^\mu \gamma^\nu = g^\mu\nu - 2iS^\mu\nu$ (where $g^\mu\nu$ is the metric tensor with diag(1, -1, -1, -1)), we obtain

$$\psi^\dagger \hat{M}^\mu\nu \psi = e\psi(x^\mu \gamma^\nu - x^\nu \gamma^\mu)\psi / 2 =$$

$$\frac{e}{2m} \psi(\hat{L}^\mu\nu + 2S^\mu\nu)\psi$$

where $\hat{L}^\mu\nu = x^\mu \hat{p}^\nu - x^\nu \hat{p}^\mu$ is the usual 4D tensor of orbital angular momentum, $S^\mu\nu$ the 4D spin tensor.

Using Eqs.(16) and (17), we obtain

$$\psi(2S^\mu\nu)\psi = \psi(\hat{L}_{\text{ms}}^\mu\nu - \hat{E}^\mu\nu)\psi$$

Therefore, $2S^\mu\nu$, rather than $\hat{S}^\mu\nu$, is the difference between the instantaneous and the usual 4D tensor of orbital angular momentum, which implies that one can only endow $2S^\mu\nu$, instead of $\hat{S}^\mu\nu$, with the property of orbital angular momentum. As mentioned above, it is the zbw that results in the differences between all instantaneous quantities and the corresponding usual ones, therefore the zbw is the origin of the 4D spin tensor of electron, and offers such intrinsic dynamic degrees of freedom for electron.

4 Conclusions

We conclude that: 1) the zbw motions are independent of the macroscopic motions of electron, their spatial magnitudes are $m/E^2$ in the longitudinal direction while $E/2$ in the transverse direction; 2) the velocities of zbw are $\pm m/E$ in the longitudinal direction while $\pm 1$ in the transverse direction; 3) the zbw motions arise from a rapid vacuum fluctuation occurring within the Compton wavelength of electron; 4) the virtual electron-positron pairs from the vacuum fluctuation always have a vanishing velocity of center-of-mass, which implies that the macroscopic velocity of the vacuum medium vanishes in all inertial systems; 5) the zbw currents $z_c$ and $z_\perp$ form a vector triplet and have the total spin projections 0 and $\pm 1$ in the direction of the momentum of electron, respectively.

References