A Novel Rectangular Element for Piezoelectric Laminated Plates*

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Abstract  Based on the classical laminated plate theory, a novel finite element formulation is presented for modeling the static response of laminated composites containing distributed piezoelectric ceramic subjected to electric loadings. A four-node rectangular composite element with an additional voltage freedom per piezoelectric layer is implemented for the analysis. The element can predict more accurately the bending response of the structure because of its new displacement radixes. Numerical examples are performed and the calculated data compare very well with existing results in the literatures.

Key words  composites;  laminated plate;  piezoelectric material;  finite element;  actuator

Piezoelectric materials are widely used as sensors and actuators in intelligent structure applications, such as vibration suppression, shape control, noise attenuation and precision positioning[1-4]. To perform the analysis of piezoelectric intelligent structures, great efforts have been made by the research community. For example, Bayley and Anderson[5], and Lee analyzed the intelligent structures with piezoelectric actuators and sensors based on analytical approaches. Tzou and Tseng[6,7], and Keilers and Chang used finite element method for plates with integrated piezoelectric sensors and actuators[8], where the host structure and the sensor/actuator are modeled by stacking isoparametric solid elements. This approach makes the size of the problem large and requires some special techniques such as Guyan reduction to reduce the total degrees of freedom. Meanwhile, the isoparametric solid element in thin plate analysis makes shear energies excessive and the stiffness coefficients in the thickness direction higher. To overcome the problems, three internal degrees of freedom should be added, that leads to more complicated formulations.

This paper proposes a efficient finite element formulation for a composite plate with piezoelectric sensors and actuators. The developed element has fewer degrees of freedom than that proposed by Suleman and Venkayya while the accuracy remains the same[9]. By modeling the piezoelectric composite laminated plates with the four-node bending element, the problems existing in the solid element are eliminated. Using the new displacement radixes, the rectangular element can predict the response of the laminated plates more accurately. The electrical degrees of freedom are assumed linearly variable across the thickness. It should be pointed out that although the formulation is based on the classical theory, the developed element is readily extended to consider the shear deformation by using a method proposed by the authors.

1 Finite Element Formulations

Consider a laminated composite plate containing
distributed piezoelectric layers that can be either bonded to the surface or embedded within the structure as shown in Fig.1.

To derive the equations of motions for the laminated composite with piezoelectric sensors/actuators, the generalized form of Hamilton’s principle is used.

\[
\delta[\int_0^l (T - \Xi) dt + \int_0^l w dt] = 0
\]

where \( T \) is the kinetic energy, \( \Xi \) is the potential energy and \( W \) is the work done by the external mechanical loadings and/or the electric fields. The kinetic energy is defined as

\[
T = \int \frac{1}{2} \rho \dot{u}_i \dot{u}_i dV
\]

where \( \rho \) is the density of the composite material, \( \dot{u}_i \) is the velocity. The potential energy is expressed as

\[
\Xi = \int \left( \frac{1}{2}\sigma_i \epsilon_i - \epsilon_i E_i E_i - \frac{1}{2} \epsilon_i \epsilon_i E_i E_i \right) dV
\]

where \( H(\epsilon_i, E_i) \) is the electric enthalpy, \( \sigma_i \) is the stress, \( \epsilon_i \) is the strain, \( E_i \) is the electric field, \( \epsilon \) is the piezoelectric constant, \( \epsilon^\sigma \) is the dielectric permittivity constant of piezoelectrical materials under constant stress field. For the non-piezoelectrical materials, the last two terms in Eq.(3) are always zero.

1.1 Piezoelectric Constitutive Equations

For polarization piezoelectrics, the properties are defined relative to the poling direction through the thickness and the material is assumed approximately isotropic in the other two directions. In matrix form, the equations governing the material properties can be written as\[5\]

\[
\epsilon = C \sigma + dE
\]

and

\[
D = d \sigma + \xi^E E
\]

where \( C \) is the elastic stiffness matrix of piezoelectric material, \( D \) is the electric displacement vector. The other symbols have the same meaning with Eq.(3) except the matrix forms here.

1.2 Generalized Strain-Displacement Relations

To analyze the deformations, a four-node rectangular composite element with an additional voltage freedom per piezoelectric layer is implemented based on the classical laminated plate theory. For the piezoelectric, the assumptions that the voltage is applied only on the polarization direction (the thickness direction) and the gradients of voltage along \( x \) and \( y \) directions are zero are accepted.

For a plate element with piezoelectric layers, the extended displacements can be written as

\[
\delta = \{\phi^e, \phi^p\}^T
\]

where \( \phi^e \) is the element node displacement vector

\[
\phi^e = \left[u^0_i, v^0_i, w^0_i, \theta_{x}, \theta_{y}\right]^T
\]

and \( \phi^p \) is the element potential vector

\[
\phi^p = \left[\phi_1, \phi_2, \ldots, \phi_j\right]^T
\]

In the above two equations, \( u^0_i, v^0_i \) and \( w^0_i \) are element node displacements in the middle plane of the composite plate, \( \theta_{x}, \theta_{y} \) are representing the rotations of the normal to the middle plane corresponding to the slope of the lamina, \( m \) is the number of the piezoelectric layer of the element.

For the rectangular plate element with four nodal points, the displacement shape functions are

\[
u = u_0 - \frac{z w}{\partial y} = \sum_{i=1}^{4} N_i u_i - 2 \sum_{i=1}^{4} \frac{\partial f_{hi}}{\partial y} \delta_{hi}
\]

\[
v = v_0 - \frac{z w}{\partial x} = \sum_{i=1}^{4} N_i v_i - 2 \sum_{i=1}^{4} \frac{\partial f_{hi}}{\partial x} \delta_{hi}
\]

\[
w = \sum_{i=1}^{4} f_{hi} \delta_{hi}
\]

where \( \delta_{hi} \) is the plate bending displacement vector

\[
\delta_{hi} = [w_i, \theta_{xi}, \theta_{yi}]^T
\]

\( N_i \) is the usual shape functions for the in-plane displacements, and \( f_{hi} \) is the proposed shape functions for the deflection, namely

\[
f_{hi} = \left[f_i, f_{mi}, f_{pi}\right]^T
\]

\[
f_i = \frac{1}{16} (4 - 2 \xi \eta - 2 \xi \eta + \xi \eta \xi \eta) (\xi + \xi \eta + \eta + \eta \xi)^2
\]

\[
f_{mi} = \frac{\eta}{32} (4 \xi^2 - 4 - \eta \eta - \eta \xi + \eta \xi + \eta^2 - \xi^2 \eta^2) \times
\]

\[
(\xi + \xi \eta + \eta + \eta \xi)^2
\]

\[
f_{pi} = \frac{\eta}{32} (4 \xi^2 - 4 - \eta \eta - \eta \xi + \eta \xi + \eta^2 - \xi^2 \eta^2) \times
\]

\[
(\xi + \xi \eta + \eta + \eta \xi)^2
\]
It is important to note that the proposed shape functions contain all sixteen poly-nominal terms up to the third order. The resulting element is a non-conforming element passed the patch test. It will be shown later that the element formulated with the proposed shape function can model the piezoelectric composite more accurately than the ACM element, although the total degrees of freedom are the same for both elements.

Base on the mechanical geometrical equations and electrostatic equations, the strain-displacement and electric field-potential relations for the element can be written as

\[
\begin{bmatrix}
\varepsilon_x^0 & \varepsilon_y^0 & \gamma_{xy}^0 & k_x & k_y & k_{xy} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
\phi_x & \phi_y & \phi_\theta \\
\phi_x & \phi_y & \phi_\theta
\end{bmatrix} = B^e \phi
\]

(17)

\[
\begin{bmatrix}
E_x & E_y & E_3 \\
0 & 0 & 0
\end{bmatrix} = -B^e \begin{bmatrix}
\phi_x & \phi_y & \phi_\theta \\
\phi_x & \phi_y & \phi_\theta
\end{bmatrix}
\]

(18)

where \( \varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0 \) are normal strains and shearing strain in the middle plane, \( k_x, k_y, k_{xy} \) are bending curvatures and twist curvature, \( B^e_x \) and \( B^e_y \) are element displacement geometrical matrix and element potential geometrical matrix, respectively.

\[
B^e_x = \begin{bmatrix}
N^0_x & 0 & 0 & 0 & 0 \\
0 & N^0_y & 0 & 0 & 0 \\
0 & 0 & N^0_{xy} & 0 & 0 \\
0 & 0 & -f_{ix,xx} & -f_{ix,yy} & -f_{ix,xy} \\
0 & 0 & -f_{ix,yy} & -f_{ix,xy} & -f_{ix,yy}
\end{bmatrix}
\]

\[
B^e_y = \begin{bmatrix}
1/h_j & 0 & 0 \\
0 & \cdots & 0 \\
0 & 0 & 1/h_j
\end{bmatrix} 
\]

(19)

where \( h_j \) is the thickness of the \( j \)th piezoelectric layer. \( B^e_y \) is a diagonal matrix under the assumption that the voltage is applied only on the polarization direction of the piezoelectric layer.

1.3 Motion Equations

Substituting Eqs.(4), (5) and Eqs.(17), (18) into Eq.(1) yields the element mass matrix, stiffness matrix, piezoelectric coupling and piezoelectric dielectric matrices as

\[
M^e = \int \int \int \rho(N^e) \hat{v} N^e \, dv
\]

(21)

\[
K^e = \int \int \int (B^e)^T D^e B^e \, dv
\]

(22)

\[
K_{ew} = \int \int \int (B^e)^T S^e B^e \, dv
\]

(23)

\[
K_{ov} = \int \int \int (B^e)^T e^o B^e \, dv
\]

(24)

The equations of motion for a piezoelectrically coupled electromechanical composite plate can be expressed as

\[
\sum_{n=1}^{N} M^e u^{m} + \sum_{n=1}^{N} K_{ew}^m u + \sum_{n=1}^{N} K_{ov}^m \phi = F
\]

(25)

\[
\sum_{n=1}^{N} K_{ew}^m u + \sum_{n=1}^{N} K_{ov}^m \phi = Q
\]

(26)

2 Numerical Applications

To demonstrate the performance of the element developed herein, a computer program is developed. Several examples with existing solutions in the literature are studied by using the proposed finite element.

2.1 Piezoelectric Bimorph

The first validation case is a piezoelectric bimorph shown in Fig.2, first experimentally investigated by Tzou and Tseng [7]. The bimorph, composed by bonding two PVDF layers together and polarization in opposite directions, is fixed at one end during the experiment. The material properties can refer to Ref.[7]. Five proposed elements are used to model the cantilever bimorph. Deflections(\( w \)) under 1 V applied voltage between the top and bottom surfaces are listed in Tab.1. The results by theoretical solution, experiment, and by triangular shell finite element and ACM model are also listed in the table [7,9,10]. It can be seen that the agreement between the present results and alternative finite element and theory solutions is quite good.
Tab.1  Deflections produced in a unit voltage

<table>
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<tr>
<th>Position</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
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<tr>
<td>Present FE</td>
<td>0.139</td>
<td>0.553</td>
<td>1.240</td>
<td>2.210</td>
<td>3.450</td>
</tr>
<tr>
<td>Theory</td>
<td>0.138</td>
<td>0.552</td>
<td>1.240</td>
<td>2.210</td>
<td>3.450</td>
</tr>
<tr>
<td>Shell FE</td>
<td>0.124</td>
<td>0.508</td>
<td>1.160</td>
<td>2.100</td>
<td>3.300</td>
</tr>
<tr>
<td>ACM FE</td>
<td>0.150</td>
<td>0.570</td>
<td>1.260</td>
<td>2.200</td>
<td>3.420</td>
</tr>
<tr>
<td>Experimental</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>3.150</td>
</tr>
</tbody>
</table>

Fig.3 shows the deflections under the same voltage with different element numbers. It proves that the present element has excellent convergence property.

Using charge Eq.(26), the sensing voltage distribution of the bimorph beam for a prescribed deflection can be obtained. The element voltages for an imposed tip deflection of 1 cm are shown in Fig.4. It can be observed a good agreement between the element voltages obtained by the proposed element with Q9-HSDT9P model in Ref.[11]. In Tzou’s theoretical work, that the integrated effect of the piezoelectric sensor is neglected results in continuous voltage output along the beam length.

2.2 Piezoelectric Composite Plate

Another example is considered to compare the numerical results obtained by the proposed finite element with the experimental data[12]. A cantilever laminated composite plate \([30_2/0]\), shown in Fig.5, is considered. Deflections of the plate are measured by using the proximity sensors. Material properties for the host composite can refer to Ref.[12].

Fig.6 shows the comparison of the deflections of the composite plate between the predictions and the test results when an electric field of 472 V/m is applied on the top piezoelectric actuators and a \(-472 V/m\) field is applied on the bottom piezoelectric actuators. Fig.6 indicates the out-of-plane longitudinal bending deformation \(W_L\) as a function of the position, where the longitudinal bending deformation \(W_L\) is defined by

\[
W_L = \frac{M}{C}
\]

(27)

where \(C\) is the width of the plate, \(M\) is the lateral...
deflection measured by the sensors shown in Fig.5. In Fig.6, the solid line is the prediction based on the finite element model developed herein and circular symbols represent the experimental data. It can be seen that the prediction agrees well with the experimental data.

3 Closing Remarks

A novel simple rectangular finite element is proposed for analyzing the mechanical-electrical response of fiber reinforced laminated composite with distributed piezoelectric sensors and/or actuators. Detailed derivations are given and numerical examples are performed. It is shown that the predictions agree very well with the experimental data and known solutions in the literatures. The element has excellent convergence property and accuracy. It should be mentioned that the element could be extended to consider the shear deformations without any difficulty.

References


Brief Introduction to Author(s)

ZHOU Yong (周勇) was born in 1976. He received his Ph.D. degree from NUAA in 2004. His research interests include: smart materials and structures, mechanical and electrical engineering.

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