On the Form Invariance and Lie Symmetry of Birkhoffian Systems

GU Shu-long, ZHANG Hong-bin
(1. Department of Physics, Chaohu College Anhui Chaohu 238000 China; 2. Shanghai Institute of Applied Mathematics and Mechanics Shanghai 200072 China)

Abstract The Lie symmetry is an invariance of the differential equations of motion under the infinitesimal transformations. The form invariance is an invariance of the form of the mechanical equations of motion under the infinitesimal transformations. In this paper, the relation between form invariance and Lie symmetry of Birkhoffian systems is presented, and the relationship unveil that the form invariance does not always mean Lie symmetry, vice versa.

Key words analytical mechanics; Birkhoffian systems; form invariance; Lie symmetry

The conservation laws (of first integrals) of mechanical systems are always of mathematical importance, and they are regarded as a manifestation of some profound physical principle. The modern means of finding conservation laws are mainly Noether symmetry and Lie symmetry. The Noether symmetry is an invariance of Hamiltonian action under the infinitesimal transformations of time and the coordinates. The Lie symmetry is an invariance of the differential equations of motion under the infinitesimal transformations. Recently, Mei presents a new invariance which is called form invariance\(^{[1-4]}\). The form invariance is an invariance of the form of the mechanical equations of motion under the infinitesimal transformations. The form invariance is different from either Noether symmetry or Lie symmetry\(^{[5,6]}\).

The study of dynamics of Birkhoffian systems is a relatively new direction of development in mathematical physics, and can be applied to quantum mechanics, biological physics, mechanics of space flight and some fields in modern engineering science\(^{[7-9]}\). In the past decade, the symmetries of Birkhoffian systems have been studied, and some important results have been obtained\(^{[10-18]}\): that of finding the conserved quantity from a Lie symmetry and that of finding the Lie symmetry from a known integral etc. The relation between form invariance and Lie symmetry of Birkhoffian systems is further studied in this paper.

1 On the Form Invariance and Lie Symmetry of Free Birkhoffian Systems

The free Birkhoffian equations can be written in the form\(^{[8]}\)

\[
\left(\frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \dot{a}^\nu - \frac{\partial B}{\partial a^\nu} - \frac{\partial R_\mu}{\partial t} = 0 \quad \mu, \nu = 1, 2, \cdots, 2n
\]  

(1)

where the function \( B = B(t, a) \) is called a Birkhoffian, \( R_\mu = R_\mu(t, a) \) are called Birkhoff functions, and \( a^\nu \) are variables. The system described by Eq.(1) is called a free Birkhoffian system.

Introduce the infinitesimal transformations of time and coordinates as

\[
t^\nu = t + \Delta t \quad a^\nu(t^\nu) = a^\nu(t) + \Delta a^\nu \quad \mu = 1, 2, \cdots, 2n
\]  

(2)

or their expansion formulae...
where \( \varepsilon \) is an infinitesimal parameter, and \( \xi_0, \xi_\mu \) are called infinitesimal generators under Eq.(3), the function \( B(t,a) \) becomes \( B(t',a') \), and \( R_\mu(t,a) \) become \( R_\mu(t',a') \).

**Definition 1** Under the infinitesimal transformation Eq.(3), if the free Birkhoffian Eq.(1) keep their form invariant, i.e.

\[
\left( \frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\mu^*}{\partial a^\nu} \right) \dot{a}^\nu - \frac{\partial B}{\partial a} = 0 \quad \mu, \nu = 1, 2, \ldots, 2n
\]

(4)

then this kind of invariance is called a form invariance of free Birkhoffian system.

The differential operator of the infinitesimal generator is

\[
X^{(0)}(t) = \xi_0 \frac{\partial}{\partial t} + \xi_\mu \frac{\partial}{\partial a^\mu} \quad k = 1, 2, \ldots, 2n
\]

(6)

and its extensions are

\[
X^{(0)}(t) = X^{(0)}(t) + (\dot{\xi}_0 - \dot{\xi}_0) \frac{\partial}{\partial a^\mu} \quad k = 1, 2, \ldots, 2n
\]

(7)

expanding \( B' \) and \( R_\mu^* \), obtain

\[
B' = B(t,a) + \varepsilon \left[ X^{(0)}(B) \right] + O(\varepsilon^2)
\]

(8)

\[
R_\mu^* = R_\mu(t,a) + \varepsilon \left[ X^{(0)}(R_\mu) \right] + O(\varepsilon^2) \quad \mu = 1, 2, \ldots, 2n
\]

(9)

From Eqs.(4)–(9), the following are obtained.

**Criterion 1** For a free Birkhoffian system expressed by Eq.(1), if the infinitesimal generators \( \xi_0 \) and \( \xi_\mu \) satisfy the following relations

\[
\left[ \frac{\partial}{\partial a^\mu} X^{(0)}(R_\mu) - \frac{\partial}{\partial a^\mu} X^{(0)}(R_\mu) \right] \dot{a}^\mu = \frac{\partial}{\partial a^\nu} X^{(0)}(B) - \frac{\partial}{\partial a^\nu} X^{(0)}(R_\mu) = 0 \quad \mu, \nu = 1, 2, \ldots, 2n
\]

(10)

then it is form invariance under the infinitesimal transformation Eq.(3).

**Proof** Substituting Eqs.(8) and (9) into Eqs.(4), using Eq.(1), and neglecting \( \varepsilon^2 \) and the higher infinitesimal terms, Eq.(10) will be obtained.

The basic idea of the Lie symmetry is to keep Eq.(1) invariant under infinitesimal transformation Eq.(3). Eq.(1) is rewritten as

\[
G(t,a,\dot{a}) = \left( \frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial R_\mu^*}{\partial a^\nu} \right) \dot{a}^\nu = 0 \quad \mu, \nu = 1, 2, \ldots, 2n
\]

(11)

**Definition 2** If the differential Eq.(11) remain invariant under infinitesimal transformation Eq.(3), i.e.

\[
G(t,a,\dot{a}) = 0
\]

(12)

then the invariance is called the Lie symmetry of the equations of motion of free Birkhoffian system.

Expanding \( G \), we have

\[
G(t,a,\dot{a}) = G(t,a,\dot{a}) + \varepsilon X^{(1)}(G) + O(\varepsilon^2)
\]

(13)

**Criterion 2** For the free Birkhoffian Eq.(1), if the infinitesimal generators \( \xi_0 \) and \( \xi_\mu \) satisfy the following relations

\[
X^{(1)}(G) = 0
\]

(14)

then the invariance is the Lie symmetry of free Birkhoffian systems.

**Proof** Substituting Eqs.(11) and (13) into Eq.(12), Eq.(14) is obtained.

From the deduction of Eqs.(4) and (12), it can be
seen that the form invariance is generally different from the Lie symmetry. For seeking their relations, we may expand the left-hand side of the Eq.(10).

\[
\left[ \frac{\partial X^{(0)}}{\partial a^\mu} - \frac{\partial X^{(0)}}{\partial a^\nu} \right] \frac{\partial}{\partial t} = -\frac{\partial}{\partial a^\mu} \frac{\partial X^{(0)}}{\partial a^\nu} - \frac{\partial}{\partial a^\nu} \frac{\partial X^{(0)}}{\partial a^\mu} \frac{\partial}{\partial t} = 0
\]

\[
\left[ \frac{\partial}{\partial a^\mu} \left( \xi \frac{\partial R_\alpha}{\partial t} + \bar{\zeta} \frac{\partial R_\beta}{\partial t} \right) - \frac{\partial}{\partial a^\nu} \left( \bar{\xi} \frac{\partial R_\alpha}{\partial t} + \bar{\zeta} \frac{\partial R_\beta}{\partial t} \right) \right] \frac{\partial}{\partial t} = 0
\]

\[
\frac{\partial}{\partial a^\mu} \left( \xi \frac{\partial R_\alpha}{\partial t} + \bar{\zeta} \frac{\partial R_\beta}{\partial t} \right) - \frac{\partial}{\partial a^\nu} \left( \bar{\xi} \frac{\partial R_\alpha}{\partial t} + \bar{\zeta} \frac{\partial R_\beta}{\partial t} \right) = 0
\]

The relation between the form invariance and the Lie symmetry is given by Eq.(15) and then the following are obtained.

**Proposition 1** For the free Birkhoffian systems, if the equations of motion are form invariance under infinitesimal transformation Eq.(3), and the following relations hold

\[
\left[ \frac{\partial}{\partial a^\mu} \frac{\partial R_\alpha}{\partial t} + \frac{\partial}{\partial a^\nu} \frac{\partial R_\beta}{\partial t} \right] \frac{\partial}{\partial t} = 0
\]

\[
\frac{\partial}{\partial a^\mu} \frac{\partial R_\alpha}{\partial t} - \frac{\partial}{\partial a^\nu} \frac{\partial R_\beta}{\partial t} = 0 \quad \mu, \nu, k = 1, 2, \cdots, 2n
\]

then the equations are also Lie symmetrical.

Let the sum of the coefficient of the term which explicitly contain \( \dot{a}^\nu \) and one don’t explicitly contain \( \ddot{a}^\nu \) equals null respectively, then

\[
\begin{align*}
\frac{\partial}{\partial a^\mu} \frac{\partial R_\alpha}{\partial t} + \frac{\partial}{\partial a^\nu} \frac{\partial R_\beta}{\partial t} &= 0 \\
\frac{\partial}{\partial a^\mu} \frac{\partial R_\alpha}{\partial t} - \frac{\partial}{\partial a^\nu} \frac{\partial R_\beta}{\partial t} &= 0 \\
\frac{\partial}{\partial a^\nu} \frac{\partial R_\alpha}{\partial t} - \frac{\partial}{\partial a^\mu} \frac{\partial R_\beta}{\partial t} &= 0
\end{align*}
\]

\[\nu = i : i \in (1, 2, \cdots, 2n) \quad \mu = 1, 2, \cdots, 2n\]

\[\mu, \nu, k = 1, 2, \cdots, 2n\]

\[\begin{align*}
\lambda_{\mu} \ddot{a}^\mu &= \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\mu} \\
\lambda_{\beta} \dot{a}^\beta &= \frac{\partial B}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\beta}
\end{align*}\]

\[\begin{align*}
\lambda_{\mu} \ddot{a}^\mu &= \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\mu} \\
\lambda_{\beta} \dot{a}^\beta &= \frac{\partial B}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\beta}
\end{align*}\]

2 On the Form Invariance and Lie Symmetry of Constrained Birkhoffian Systems

Suppose that the variables \( a^\mu \) of the systems are not independent of each other, but are restricted by some constraints, for which the systems are called the constrained Birkhoffian systems. If the restrictions can be expressed as the following constraint equations

\[f_\beta(t, a) = 0 \quad \beta = 1, 2, \cdots, g\]

According to Eqs.(1) and (19), using the method of Lagrange’s multiplier, the equations of motion with multipliers of the constrained Birkhoffian systems are obtained.

\[\begin{align*}
\Omega_{\mu} \ddot{a}^\mu &= \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\mu} \\
\lambda_{\beta} \dot{a}^\beta &= \frac{\partial B}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\beta}
\end{align*}\]

\[\begin{align*}
\lambda_{\mu} \ddot{a}^\mu &= \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\mu} \\
\lambda_{\beta} \dot{a}^\beta &= \frac{\partial B}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\beta}
\end{align*}\]

where

\[\Omega_{\mu} = \frac{\partial R_\mu}{\partial a^\nu} \quad \mu, \nu = 1, 2, \cdots, 2n\]

which is the Birkhoff tensor. Before integrating the equations of motion, \( \lambda_{\mu} \) as function of \( t, a \) can be
expressed as

\[ \lambda_\beta = \frac{A_\beta}{A} \left[ \frac{\partial f_\beta}{\partial a^\nu} \Omega^{\nu \rho \mu} \left( \frac{\partial B}{\partial a^\rho} + \frac{\partial R_\mu}{\partial t} \right) + \frac{\partial f_\beta}{\partial t} \right] \]

\[ \mu, \rho = 1, 2, \ldots, 2n \quad \beta, \gamma = 1, 2, \ldots, g \]

(22)

where

\[ A = \det \left( \frac{\partial f_\beta}{\partial a^\nu} \Omega^{\nu \rho \mu} \right) \]

\[ \mu, \rho = 1, 2, \ldots, 2n \quad \beta, \gamma = 1, 2, \ldots, g \]

(23)

\( A_\beta \) is an algebraic complement of element \((\lambda, \beta)\) in the determinant \(A\), \(\Omega^{\nu \rho \mu}\) is the Birkhoff conserved tensor.

Let

\[ P_\mu = P_\mu(t, a) = \lambda_\beta \frac{\partial f_\beta}{\partial a^\nu} \]

\[ \mu = 1, 2, \ldots, 2n \quad \beta = 1, 2, \ldots, g \]

(24)

where \( \lambda_\beta \) is expressed by Eq.(22), substituting Eq.(24) into Eq.(20), we obtain

\[ \left( \frac{\partial R_\mu}{\partial a^\rho} - \frac{\partial R_\rho}{\partial a^\nu} \right) \right. \frac{\partial a^\nu}{\partial t} - \frac{\partial B}{\partial a^\rho} - \frac{\partial R_\mu}{\partial t} = P_\mu \]

\[ \mu, \nu = 1, 2, \ldots, 2n \]

\[ \frac{\partial f_\beta}{\partial a^\nu} \Omega^{\nu \rho \mu} \left( \frac{\partial B}{\partial a^\rho} + \frac{\partial R_\mu}{\partial t} \right) + \frac{\partial f_\beta}{\partial t} = \]

(25)

Eq.(25) is called the constrained Birkhoffian equations, which have no the constraint multipliers \( \lambda_\beta \).

**Definition 3** If the form of constrained Birkhoffian Eq.(25) and the constraint Eq.(19) remain invariant under the infinitesimal transformation Eq.(3), i.e.

\[ f_\beta^* = f_\beta(t^*, a^*) = 0 \quad \beta = 1, 2, \ldots, g \]

(26)

\[ \left( \frac{\partial R_\mu^*}{\partial a^\rho} - \frac{\partial R_\rho^*}{\partial a^\nu} \right) \right. \frac{\partial a^\nu}{\partial t} - \frac{\partial B^*}{\partial a^\rho} - \frac{\partial R_\mu^*}{\partial t} = P_\mu^* \]

\[ \mu, \nu = 1, 2, \ldots, 2n \]

(27)

then this kind of invariance is called a form invariance of the constrained Birkhoffian systems.

Expanding \( P_\mu^* \) and \( f_\beta^* \), obtain

\[ P_\mu^* = P_\mu(t, a) + \varepsilon \left[ X^{(0)}(P_\mu) \right] + O(\varepsilon^2) \quad \mu = 1, 2, \ldots, 2n \]

(29)

\[ f_\beta^* = f_\beta(t, a) + \varepsilon \left[ X^{(0)}(f_\beta) \right] + O(\varepsilon^2) \quad \beta = 1, 2, \ldots, g \]

(30)

From Eqs.(6)~(9), (26), (27), (29) and (30), the following are obtained.

**Criterion 3** For the constrained Birkhoffian systems expressed by Eqs.(1) and (19), if the infinitesimal generators \( \xi \) and \( \xi \) satisfy the following relations

\[ X^{(0)} \left[ f_\beta(t, a) \right] = 0 \quad \beta = 1, 2, \ldots, g \]

(31)

\[ \left[ \frac{\partial}{\partial a^\rho} X^{(0)}(R_\mu) - \frac{\partial}{\partial a^\nu} X^{(0)}(R_\mu) \right] \right. \frac{\partial a^\nu}{\partial t} - \frac{\partial a^\rho}{\partial t} X^{(0)}(B) = \]

(32)

then it is form invariant under the infinitesimal transformation Eq.(3).

**Proof** Substituting Eq.(30) into Eq.(31), using Eq.(19), substituting Eqs.(8), (9) and (29) into Eq.(32), using Eq.(1), and neglecting \( \varepsilon \) and the higher infinitesimal terms, Eqs.(31) and (32) will be obtained.

Analogous above the discussion, Eq.(25) is rewritten as

\[ H(t, a, \dot{a}) = \left( \frac{\partial R_\mu^*}{\partial a^\rho} - \frac{\partial R_\rho^*}{\partial a^\nu} \right) \right. \frac{\partial a^\nu}{\partial t} - \frac{\partial B^*}{\partial a^\rho} - \frac{\partial R_\mu^*}{\partial t} = P_\mu^* \]

\[ \mu, \nu = 1, 2, \ldots, 2n \]

(33)
Definition 4 If the constrained Eq.(19) and the differential Eq.(25) remain invariant under infinitesimal transformation Eq.(3), i.e. Eq.(31) hold and

\[ H(t^*, a^*, \dot{a}^*) = 0 \] (34)

then the invariance is called the Lie symmetry of the equations of motion of constrained Birkhoffian system.

Expanding \( H \), obtain

\[ H(t^*, a^*, \dot{a}^*) = H(t, a, \dot{a}) + \varepsilon X^{(1)}(H) + O(\varepsilon^2) \] (35)

Criterion 4 For the constrained Birkhoffian Eqs.(1) and (19), if the infinitesimal generators \( \xi_\mu \) and \( \xi_\nu \) satisfy Eq.(31) and the following relations

\[ X^{(1)}(H) = 0 \] (36)

then the invariance is the Lie symmetry of the constrained Birkhoffian systems.

Proof Substituting Eqs.(33) and (35) into Eq.(34), Eq.(36) is obtained.

For the constrained Birkhoffian systems, the relation between the form invariance and Lie symmetry can be sought as

\[
\left[ \frac{\partial X^{(1)}(R_\mu)}{\partial a^\mu} - \frac{\partial X^{(0)}(R_\nu)}{\partial a^\nu} \right] a^\nu - \frac{\partial X^{(0)}(B)}{\partial a^\nu} \frac{\partial X^{(0)}(R_\mu)}{\partial t} - \\
X^{(1)}(P_\mu) = \left[ \frac{\partial}{\partial a^\mu} \left( \xi_\mu \frac{\partial R_\mu}{\partial t} + \xi_\nu \frac{\partial R_\nu}{\partial a^\nu} \right) - \frac{\partial}{\partial a^\nu} \left( \xi_\mu \frac{\partial R_\mu}{\partial t} + \xi_\nu \frac{\partial R_\nu}{\partial a^\nu} \right) \right] a^\nu - \frac{\partial}{\partial a^\nu} \left( \xi_\mu \frac{\partial B}{\partial t} + \xi_\nu \frac{\partial B}{\partial a^\nu} \right) - \frac{\partial}{\partial t} \left( \xi_\mu \frac{\partial R_\mu}{\partial t} + \xi_\nu \frac{\partial R_\nu}{\partial a^\nu} \right) - \\
\xi_\mu \frac{\partial P_\mu}{\partial a^\nu} - \left( \xi_\mu \frac{\partial P_\mu}{\partial t} + \xi_\nu \frac{\partial P_\nu}{\partial a^\nu} \right) = X^{(1)}(H(t, a^*, \dot{a}^*)) + \\
\left[ \frac{\partial \xi_\mu}{\partial a^\mu} \frac{\partial R_\mu}{\partial t} + \frac{\partial \xi_\nu}{\partial a^\nu} \frac{\partial R_\nu}{\partial a^\nu} \right] a^\nu - \frac{\partial \xi_\mu}{\partial a^\nu} \frac{\partial R_\mu}{\partial t} - \frac{\partial \xi_\nu}{\partial a^\mu} \frac{\partial R_\nu}{\partial a^\nu} - \\
\xi_\mu \left( \frac{\partial R_\mu}{\partial a^\mu} + \frac{\partial R_\nu}{\partial a^\nu} \right) a^\nu - \frac{\partial \xi_\mu}{\partial a^\nu} \frac{\partial B}{\partial t} - \frac{\partial \xi_\nu}{\partial a^\mu} \frac{\partial B}{\partial a^\nu} - \frac{\partial \xi_\nu}{\partial a^\mu} \frac{\partial R_\mu}{\partial a^\nu} - \\
\frac{\partial \xi_\mu}{\partial t} \frac{\partial R_\mu}{\partial a^\nu} - \frac{\partial \xi_\mu}{\partial a^\nu} \frac{\partial R_\nu}{\partial a^\nu} \right] \mu, \nu, k = 1, 2, \cdots, 2n
\] (37)

The relation between the form invariance and the Lie symmetry is given by Eqs.(15) and then the following are obtained.

Proposition 2 If the constrained Birkhoffian Eqs.(1) and (19) are form invariance under infinitesimal transformation Eq.(3), and the following relations hold Eq.(16), then the systems are also Lie symmetrical.

Obviously, for Birkhoffian systems, the constraint has no effect on the relation between the form invariance and Lie symmetry.

3 An Illustrative Example

Le’s study the form invariance and Lie symmetry of a second order Birkhoffian systems as follows

\[ R_1 = 0 \quad R_2 = ta^1 \quad B = \frac{1}{2} [ (ta^1)^2 + (a^2)^2 ] \] (38)

From Eq.(10), get

\[ \frac{\partial}{\partial a^1} (\xi_1 a^1 + \xi_2 t) = 0 \] (39)

\[ \frac{\partial}{\partial a^1} [ (\xi_1 t a^1)^2 + \xi_2 t^2 a^1 + \xi_3 a^2 ] = 0 \] (40)

verify

\[ \xi_0 = t \quad \xi_1 = -\frac{a^1}{t} \quad \xi_2 = 0 \] (42)

\[ \xi_0 = t \quad \xi_1 = -\frac{a^1}{t} \quad \xi_2 = 0 \] (43)

are two sets of solutions of the Eqs.(39), (40). But neither of them satisfies the Eqs.(17) nor (18), so they are not Lie symmetry.

References


5 Conclusion

The original audio signal is decomposed by DWT into three layers before the watermark signal is embedded into the coefficients, then the watermarked audio signal is obtained via DWT reconstruction. Simulation shows that all of the watermarking embedded in four kinds of music are blind to HAS and are fairly robust. The proposed algorithm is relatively simple and unlike the algorithm in Ref.[3], our detection process does not need the original audio signal. We choose orthonormal wavelets for simplicity, but biorthogonal wavelets and wavelet packets may be used on the same idea. In our future study, we will investigate these possible extensions of our algorithm.

References


Brief Introduction to Author(s)

FU Yu (傅瑜) was born in 1962. He received the Ph.D. degree from Xidian University in 1998. He is now an Associate Professor in Zhongshan College, UESTC. His research interests include: wavelets and their application, digital watermarking and numerical analysis.

WANG Bao-bao (王宝宝) was born in 1963. He obtained the Ph.D. degree from Xidian University in 1999. He is now an Associate Professor at Xidian University. His research interests include: multimedia data processing and other related areas.