Lyapunov Criteria for Structural Stability of Supply Chain System

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Abstract In this paper, based on Cobb-Douglas production function, the structural stability of the supply chain system are analyzed by employing Lyapunov criteria. That the supply chain system structure, with the variance of the rate of re-production input funding, becomes unstable is proved. Noticeably, the solutions shows that when the optimal combination of input parameter element, the qualitative properties of supply chain system change and the supply chain system becomes unstable.

Key words the supply chain system; the structural stability; Lyapunov characteristic number

The supply chain is a value chain including material supply, production manufacturing, distributor and retailer and customer, and it is a process which begins with the market demand, and provides customer with production and service[1]. The supply chain management is a high-integrated management model, which is a multi-enterprise or multi-segment management, and it integrates of material, production, and inventory management and according to the clues of logistics flows, fund flows and information flows.

The structural stability of the supply system influences its performance greatly[2]. The concept of structural stability is basically motivated by the application to the supply chain behavior. The structural stability of the supply chain is a crucial issue in supply chain management. In recent years, scholars from all over the world have adopted a system integration modeling method to launch their researches in the field of supply chain management and provide us with a series of integrated model of supply chain system[3~6]. However, most of them are based on the analysis on certain condition and limited by a fixed or determinable channel distribution, it is difficult to identify the characters of the system structure during modeling integrated supply chain system.

The supply chain system involves many node firms. Those can be single production-inventory system composing complicated multi-echelon production-inventory system as the technique and management environment changes. When there is a network structure in the supply chain system, the uncertainty is rapidly dispersed in the production network. If deviation exceeds allow value, the supply chain system, state and behavior would have been deviated under perturbations of a kind of factors of interior or environment. Moreover, the deviation increased, the production process give birth to fluctuation, the supply chain system is unstable. The stability research of the integrated supply chain system is very important. However, the literatures on analyzing the stability of integrated supply chain system are few.

A Lyapunov criterion of structural stability for a set of dynamic system by means of the Lyapunov characteristic number is a useful method. The Lyapunov criteria is that when all of the Lyapunov characteristic numbers of the system are less than zero, the dynamic system is stable; if one Lyapunov characteristic number is greater than zero in the system, the dynamic system becomes unstable. This criterion has the merits of smaller computation and simpler searching method as compared with the method based on the Floquet theory, which needs to search all characteristic values of matrix of basic solutions of the dynamic system when the system is of high dimension.

In this paper, an integrated model of the supply

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chain based on Cobb-Douglas production function is presented and the structural stability the supply chain system are studied by employing Lyapunov criteria. The main purpose of this paper is to provide a general a new insight of the study on the supply chain management system.

1 Dynamic Integrated Model Based on Cobb-Douglas Production Function

The state equation of the inventory for components supplier is as follows

\[ x_{i+1} = x_i + c_i - v_i \quad t = 1, 2, \ldots, N \]  

where \( x_i \) is the components inventory of supplier in the period \( t \); \( c_i \) is the components production of supplier in the period \( t \); \( v_i \) is the offering amount of components of manufacturer in the period \( t \).

The state equation of the inventory for manufacturer is

\[ y_{i+1} = y_i + p_i - s_i \quad t = 1, 2, \ldots, N \]  

where \( y_i \) is the inventory of manufacturer in the period \( t \); \( p_i \) is the output of manufacturer in the period \( t \); \( s_i \) is the amount of manufacturer supply its distributor in the period \( t \).

Assume that the components has a fixed price \( \varphi \) in a competitive market, the fund that the amount of components of manufacturer is \( k_i = \varphi v_i \). By means of Cobb-Douglas production function\[7\], the output of manufacturer in the period \( t \) is

\[ p_i = A k_i^\alpha t_i^\beta \quad t = 1, 2, \ldots, N \]  

where \( l_i \) and \( k_i \) are the input of labor force and the input of fund, respectively. \( p_i \) is the output of the \( t \) period, and the parameters, \( \alpha > 0 \), \( \beta > 0 \). Assume that the scale profit is constant, i.e., \( \alpha + \beta = 1 \), and production prices is 1. In \( t + 1 \) period, the manufacturer must input \( rp_i \), the funding of re-production is\[8\]

\[ k_{i+1} = rp_i = r A k_i^\alpha t_i^\beta \]  

The profit margin of funding is \( p_i \), the wages rate is \( p_i \), so the input cost of manufacturer in the period \( t \) is

\[ C = p_i k_i + p_i l_i \]  

In a certain period, the social production scale is definite, and the total production competence is relatively constant. Regard the total cost as constant, so

\[ l_{i+1} = C / p_i - p_i k_{i+1} / p_i \]  

Let distributor inventory in the period \( t \) be equal to the market demand in the period \( t - 1 \), that is \( s_i = d_{i-1} \). Suppose that the inventory waster ratio is \( (1 - B) \), and shatter quantity is \( (1 - B)z_i \), then

\[ z_{i+1} = \beta z_i + d_{i-1} - d_i \quad t = 1, 2, \ldots, N \]  

where \( B \) is normal ratio, \( z_i \) is the inventory of distributor in the period \( t \). The inventory and output of the supply, manufacturing and sale in the whole supply chain are non-negative value, that is: \( x_i, y_i, z_i, p_i \geq 0 \) \( (t = 1, 2, \ldots, N) \). The dynamic model of the integrated supply chain system is

\[ \begin{align*}  
x_{i+1} &= x_i + M c_i - k_i \varphi^{-1} 
y_{i+1} &= y_i + A k_i^\alpha t_i^\beta - S p_i 
z_{i+1} &= B z_i + d_{i-1} - d_i 
k_{i+1} &= r A k_i^\alpha t_i^\beta 
l_{i+1} &= C / p_i - p_i k_{i+1} / p_i \end{align*} \]  

2 Analysis on Structural Stability of Supply Chain System Based on Lyapunov Criteria

According to the bifurcation theory of nonlinear dynamic system, the Lyapunov characteristic number carry is an units of an inverse time and give a typical time scale for the divergence or convergence of nearby trajectories, which play an important role in the investigation on the nonlinear dynamical system\[9\].

Applying the Lyapunov characteristic number is to distinguish the structural stability of dynamical system\[10\]. Therefore, The Lyapunov characteristic number is equal to the real parts of coefficients matrix eigenvalues of this system. Thus, the criteria for structural stability of supply chain system based on the Lyapunov characteristic number can be easily got.
Since the Lyapunov characteristic number could get its value by the numerical integral of dynamical system, which is relative to the numerical method of searching all the eigenvalues of the coefficients matrix of dynamical system, it is quite a simple and convenient procedure to this problem.

The structural stability of supply chain system based on Lyapunov criteria has an important application value. When the optimal combination of input parameter elements, then

Substituting Eqs. (30) into (29), thus

\[ k_0 = \frac{[C(\alpha p_i)/(\beta p_i)]/[p_i + p_i(\alpha p_i)/(\beta p_i)]}{\alpha C / p_h} = k' \]

\[ l_0 = C / [p_i + p_i(\alpha p_i)/(\beta p_i)] = \beta C / p_i = l' \]

The equilibrium point of the system is the optimal combination of input parameter elements of manufacture. Applying the linear method to the system near the origin, we get the real matrix \( J \) of system as follows

\[
J = \begin{bmatrix}
1 & 0 & 0 & -\varphi^{-1} & 0 \\
0 & 1 & 0 & \alpha A & k_{i}^{-1}l_i^{-1} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -r \alpha A & k_{i}^{-1}l_i^{-1} \\
0 & 0 & 0 & 0 & -\frac{p_x}{p_i} \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

From the matrix \( J \), we can get characteristic equation is \( |J - \lambda I| = 0 \).

Computing this determinant, we get the characteristic equation of the real matrix of system \( A \)

\[ -\lambda(\lambda - 1)^2(\lambda^2 - \lambda r A k_{i}^{-1}l_i^{-1}) + rA\beta k_{i}^{-1}l_i^{-1}(p_i / p_i) = 0 \]

simplifying

\[ \lambda(\lambda - 1)^2 \left[ \lambda^2 - \lambda r A k_{i}^{-1}l_i^{-1} \right] + rA\beta k_{i}^{-1}l_i^{-1}(p_i / p_i) = 0 \]

Which simplifies to

\[ \lambda(\lambda - 1)^2(\lambda^2 - \lambda r A k_{i}^{-1}l_i^{-1}) + rA\beta k_{i}^{-1}l_i^{-1}(p_i / p_i) = 0 \]

Further, which simplifies to

\[ rA k_{i}^{-1}l_i^{-1} = rA\alpha(C(\alpha / p_i) / (\beta / p_i)) + p_i(\alpha / p_i) / (\beta / p_i) = \alpha \]

\[ rA\beta k_{i}^{-1}l_i^{-1} = rA\beta(C(\alpha / p_i) / (\beta / p_i)) \times \left[ C / p_i + p_i(\alpha / p_i) / (\beta / p_i) \right] = \beta(\alpha / p_i) \]

Let \( \xi = p_x / p_i \), \( z = \alpha(\alpha / p_i) / (\beta / p_i) \), the characteristic
The equation of Jacobi matrix $J$ is

$$\lambda(\lambda-1)^2(\lambda^2 - \alpha \lambda + \xi)=0 \quad (17)$$

From Eq.(17), the eigenvalue of Jacobi matrix $J$ are $\lambda_1=0$, $\lambda_2=1$, $\lambda_3=-1$, $\lambda_{4,5}=\left(\alpha \pm \sqrt{\alpha^2 - 4\xi} \right) / 2$.

When $\lambda_1=0$, the supply chain system is marginally stable. When $\lambda_2=1$ the supply chain system becomes unstable, and when $\lambda_3=-1$, the supply chain system is stable.

When $\lambda_{4,5}=\left(\alpha \pm \sqrt{\alpha^2 - 4\xi} \right) / 2$, since $\alpha > 0$, $\beta > 0$, $r > 0$, then all the eigenvalues of coefficients matrix of this system have the positive real parts, the structural of supply chain system becomes unstable.

### 3 Conclusions

In this paper, based on Cobb-Douglas production function, the stability of the supply chain system structure is analyzed. The results show that the supply chain system structure, with the shocks of the rate of reproduction inputs, exceeds some certain ranges in the whole feasible space, is not stable, and the supply chain system cannot operate and directly affects the process of manufacture and the service level. Noticeable, the supply chain system structure is not stable in the condition of optimal combination of the input factors and it is obvious that almost system would not optimize state in this condition.

### References


### Brief Introduction to Author(s)

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