Construction of Solution for the Third Order Dispersion Evolution Equation*

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Abstract The Golstein’s strong solution formula of the second order evolution equation expands to that of the third dispersion equation by the analogy method. The semigroup expressions of its generating operator of the third order dispersion equation are obtained, and the expression to satisfy the semigroup conditions in the three orthogonal Hilbert space of the construction is also proved. Furthermore, the necessary and sufficient conditions of the generating operator’s unitary semigroup are given.

Key words Hilbert space; generating operator; semigroup; evolution equation

1 The Third Order Dispersion Evolution Equation

Let $E$ be a Hilbert space, $A$ be an adjoint dense defined linear operator in $E$, $I$ be a identical operator in $E$, $0 \in \rho(A) = \{ \lambda : (\lambda I - A) \text{ is a regular operator} \}$, $f(t) : [0, T] \rightarrow E$ be a strong continuous function, the second order evolution equation be

$$\frac{d^2x(t)}{dt^2} + Ax(t) = f(t), \forall t \in [0, T], x_0 \in D(A^{1/2}), (j = 0, 1),$$

and $f(t)$, $A^{1/2} f(t)$ be continuouson $[0, T]$. The Golstein’s strong solution formula of the above second order evolution equation is

$$x(t) = \cosh(tA^{1/2})x_0 + \int_0^t \cosh(t(A^{1/2})x_1 dx_1 + \int_0^t \int_0^t \cosh(t(A^{1/2}))f(t_1) dt_1 dt_2.$$  

where \( \cosh(tA^{1/2}) = (\exp(tA^{1/2}) + \exp(-tA^{1/2})) / 2. \)

Let the third order dispersion equation be expanded from above evolution equation

$$\begin{cases} \frac{d^3x(t)}{dt^3} + Ax(t) = f(t) & \forall t \in [0, T] \\ x_0 \in D(A^{1/3}) & j = 0, 1, 2 \\ f(t) \in D(A^{2j/3}) & f(t), A^{1/3} f(t) \text{ be continuouson} [0, T] \end{cases}$$

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Let the primo item of the construction of the solution for the Eq.(2) be \( S_0(t, A^{1/3}) \) and \( B = A^{1/3} \), so the solution of the Eq.(2) on analogy with the expression of the Eq.(1) is

\[
x = S_0(t, B)x_0 + \int_0^t S_0(t, B)x_1 dt_1 + \int_0^{t} \int_0^{t_1} S_0(t, B)x_2 dt_1 dt_2 + \int_0^{t} \int_0^{t_1} \int_0^{t_2} S_0(t, B)f(t_3) dt_1 dt_2 dt_3
\]

(3)

\( S_0(t, A^{1/3}) \) on analogy with \( \cosh(A^{1/2}) \), let

\[
S_0(t, B) = \frac{1}{3}(\exp(\omega_0 B) + \exp(\omega_1 t B) + \exp(\omega_2 B))
\]

\[
\omega_0 = \frac{1 + i\sqrt{3}}{2}, \quad \omega_1 = -1, \quad \omega_2 = \frac{1 - i\sqrt{3}}{2}, \quad i = \sqrt{-1}
\]

the apiece item of the Eq.(3) be calculated, then

\[
x = S_0(t, B)x_0 + \int_0^t S_0(t, B)x_1 dt_1 + \frac{1}{3} B^{-1}(\omega_0 \exp(\omega_1 t B) + \omega_1 \exp(\omega_2 t B))x_1
\]

\[
x = S_0(t, B)x_0 + \int_0^t \int_0^{t_1} S_0(t, B)x_2 dt_1 dt_2 = -\frac{1}{3} B^{-1}(\omega_0 \times \exp(t_2 \omega_0 + \omega_1 \exp(t_2 \omega_1 + \omega_2 \exp(t_2 \omega_2)))x_2
\]

so

\[
\frac{d}{dr} S_0(t, B) = -B^2 S_0(t, B)
\]

\[
\frac{dS_0(t, B)}{dt} = S_0(t, B)
\]

\[
\frac{dS_0(t, B)}{dt} = S_0(t, B)
\]

then

\[
x = S_0(t, B)x_0 + S_0(t, B)x_1 + S_0(t, B)x_2 + \int_0^t [\int_0^{t_1} S_0(t, B) f(t_1) dt_1] dt_2 dt_3
\]

\[
x = -B^3 S_0(t, B)x_0 + S_0(t, B)x_1 + S_0(t, B)x_2 + \int_0^t [\int_0^{t_1} S_0(t, B) f(t_1) dt_1] dt_2 dt_3
\]

\[
x = -B^3 S_0(t, B)x_0 + B^3 S_0(t, B)x_1 + S_0(t, B)x_2 - \int_0^t \int_0^{t_1} B^2 S_0(t, B) f(t_2) dt_1 dt_2 dt_3
\]

Let \( x = x_{11}, x_{11} = x_{12}, x_{12} = x_{31} \), then Eq.(2) turn to be

\[
\frac{d}{dr} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{31} \end{bmatrix} = \begin{bmatrix} I \\ -B^2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{31} \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}
\]

let

\[
M = \begin{bmatrix} I \\ -B^2 \end{bmatrix} F = \begin{bmatrix} 0 \\ f \end{bmatrix}
\]

so \( M \) is the generating operator of the Eq.(2), let \( M \) generate semigroup \( \{T, t \geq 0\} \), then

\[
(x_{11}, x_{12}, x_{31})' = T(t)(x_{11}, x_{12}, x_{31})' + \int_0^T t_2 F d\xi
\]

namely

\[
(x, x, x, x)' = T(t)(x_0, x_1, x_2)' + \int_0^T t_2 F d\xi
\]

then

\[
T = S_0(t, B)N_0 + S_0(t, B)N_1 + S_0(t, B)N_2
\]

\[
N_0 = \begin{bmatrix} I \\ I \\ -B^2 \end{bmatrix}, \quad N_1 = \begin{bmatrix} I \\ I \\ -B^2 \end{bmatrix}
\]

If \( \{T, t \geq 0\} \) is the semigroup which \( M \) generates in Hilbert space \( E_1 = D(B^2) \times D(B) \times E \), then above hypothesis comes into existence, and the expression of the Eq.(3) is the construction of solution for the three dispersion evolution Eq.(2).

As follows, that \( T \) is the semigroup which \( M \) generates is proved and the necessary and sufficient conditions which the \( T \) is the unitary semigroup are given.

Let \( A = \int_0^\lambda \lambda \delta E_x, \quad B = \int_0^\lambda \lambda^2 \delta E_x, \quad \{E_x : 0 \leq \varepsilon < \lambda \leq \lambda < \infty\} \) be the spectra of \( A \), then \( B \) be also an adjoint dense defined linear operator in \( E \).

1 Hilbert Space

Let \( B^m = B B^{m-1} \), \( B^0 = I \), \( m = 1,2,3 \), \( E_3 = D(B^2) \times D(B) \times E \), import an inner product in
\(E_3\)

\[\forall z_1 = (x_1, x_2, x_3) \in E_3, \quad z_2 = (y_1, y_2, y_3) \in E_3, \quad <z_1, z_2> = B^2 z_1 \cdot B z_2 + B x_1 \cdot B y_2 + x_3 y_3\]

then \(E_3\) is a Hilbert space. Let \(D(M) = D(B^2) \times D(B^2) \times D(B^2)\), \(D(M)\) be a dense set in \(E_3\).

2 Conclusion Proof

\(0 \in \rho(A), A^{-1}\) exist. \(B \exp(iB)x = \exp(iB)Bx, \forall x \in D(B); \forall t, s \geq 0, y = (x_1, x_2, x_3) \in E_3\)

\[T_{s, y} = (S_1(t, B)N_0 + S_1(t, B)N_1 + S_1(t, B)N_2) \times (S_0(s, B)N_0 + S_0(s, B)N_1 + S_0(s, B)N_2)y = (S_0(t, B)S_0(s, B) - B^3S_0(t, B)S_0(s, B) - B^3S_0(t, B)N_0)N_0 + (S_0(t, B)S_0(s, B) - B^3S_0(t, B)S_0(s, B)N_1 + (S_0(t, B)S_0(s, B) + S_0(t, B)S_0(s, B) + S_0(t, B))\]

\[S_0(t, s, B)]N_2y = \begin{bmatrix} S_0(t + s, B)N_0 + \frac{1}{9}(\omega_0 + \omega_1 + \omega_2)(\exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_0s)B) + \exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_1s)B) + \exp((\omega_0t + \omega_2s)B) + \exp((\omega_0t + \omega_1s)B) \end{bmatrix}N_2y = T_{s, y}

let \(\|\exp(iB)t\| < W \exp(tao), W > 0, \omega > 0, \text{then } \|S_0(t, B)\| \leq W \exp(tao). \text{So } \|T_{s, y}\| \leq \sqrt{6} W \exp(tao)(1 + B^{1,4}) \times \left\|T_{s, y}\right\|_{L^2}

Thus that \(\{T_{s, t} \geq 0\}\) is a semigroup in \(E_3\) is proved.

Let’s calculate the generating operator of the semigroup \(T_{s, t}\) as follows

\[T_{s, y} - y = \begin{bmatrix} S_0(t, B)x_1 - x_1 + S_1(t, B)x_2 + S_2(t, B)x_3 \cr \cr -B^3S_0(t, B)x_1 + S_1(t, B)x_2 - x_2 + S_2(t, B)x_3 \cr \cr -B^3S_0(t, B)x_1 - B^3S_2(t, B)x_2 + S_0(t, B)x_3 - x_3 \end{bmatrix}

\[\lim_{t \rightarrow 0}^1(S_0(t, B)x_1 - x_1) = \lim_{t \rightarrow 0}^1 \frac{1}{3}(\exp(taoB)x_1 - x_1 + \exp(taoB)x_1 - x_1 + \exp(0tB)x_1 - x_1) = \frac{1}{3}(\omega_0 + \omega_1 + \omega_2)Bx_1 = 0 \quad j = 1, 2, 3

\[\lim_{t \rightarrow 0}^1(S_0(t, B)x_1 - x_1) = \lim_{t \rightarrow 0}^1 -\frac{1}{3}B^3(\omega_0 \exp(taoB)x_1 - x_1 + \omega_1(\exp(taoB)x_1 - x_1) + \omega_2(\exp(taoB)x_1 - x_1)) = -\frac{1}{3}B^3(\omega_0 + \omega_1 + \omega_2)Bx_1 = 0 \quad \forall x \in E

\[\text{then } \lim_{t \rightarrow 0}^1(T_{s, y}) = (x_1, x_2, -B^3x_3) = My, D(M) \subseteq E_3

Let’s discuss the unitary characters of the semigroup \(T_{s, t}\) as follows

\[\forall y = (x_1, x_2, x_3) \in E_3, \quad <x_1, x_2, x_3> = 1 \quad <x_1, x_2, x_3> = 0 \quad \|\exp(iM)y\|_{L^2} = \|S_0(t, B)y\|_{L^2} + \|BS_0(t, B)y\|_{L^2} + \|B^2S_0(t, B)y\|_{L^2} + \|B^3S_0(t, B)y\|_{L^2} + \|\exp(iM)y\|_{L^2}^2 + \|\exp(iM)y\|_{L^2}^2 + \|\exp(iM)y\|_{L^2}^2 - \|\exp(iM)y\|_{L^2}^2 - \|\exp(iM)y\|_{L^2}^2 - \|\exp(iM)y\|_{L^2}^2

\[\frac{d}{dt} \|\exp(iM)y\|_{L^2} = \frac{B}{3}(\|\exp(iM)y\|_{L^2}^2 + \|\exp(iM)y\|_{L^2}^2 - \|\exp(iM)y\|_{L^2}^2 + \|\exp(iM)y\|_{L^2}^2)

if only if

\[\|\exp(iM)y\|_{L^2} = \sqrt{\|\exp(iM)y\|_{L^2}^2 + \|\exp(iM)y\|_{L^2}^2}

\(M\) generates the unitary semigroup \(\{T_{s, t} \geq 0\}\).
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