The $k$-Diameter of a Kind of Circulant Graph

ZHANG Xian-di

(School of Applied Mathematics, UESTC Chengdu 610054 China)

Abstract The diameter of a graph $G$ is the maximal distance between pairs of vertices of $G$. When a network is modeled as a graph, diameter is a measurement for maximum transmission delay. The $k$-diameter $d_k(G)$ of a graph $G$, which deals with $k$ internally disjoint paths between pairs of vertices of $G$, is a extension of the diameter of $G$. It has widely studied in graph theory and computer science. The circulant graph is a group-theoretic model of a class of symmetric interconnection network. Let $C_n(i, n/2)$ be a circulant graph of order $n$ whose spanning elements are $i$ and $n/2$, where $n \geq 4$ and $n$ is even. In this paper, the diameter, $2$-diameter and $3$-diameter of the $C_n(i, n/2)$ are all obtained if gcd$(n, i)=1$, where the symbol gcd$(n, i)$ denotes the maximum common divisor of $n$ and $i$.

Key words distance; diameter; $k$-distance; $k$-diameter; circulant graph

The terminology and notion in this paper are similar to Ref.[1], all graphs discussed here are finite and simple.

The diameter $d(G)$ of a graph $G$ is the maximal distance between pairs of vertices of $G$. The connectivity of $G$ is the minimum number of vertices needed to be removed in order to disconnect the graph. When a network is modeled as a graph, a vertex represents a node of processor (or a station) and an edge between two vertices is the link (or connection) between those two processors. In this context, diameter is a measurement for maximum transmission delay and connectivity is a good parameter to study how much tolerant the network can be in the occasion of node failures. Sometimes, we are interested in looking at a collection of multipaths between a pair of two vertices rather than at a single shortest path between them. So parameters of the $k$-wide distance (or $k$-distance) and $k$-wide diameter (or $k$-diameter) are introduced. They are extension of distance and diameter. Some kinds of graphs (or networks), such as cycles, complete graphs, hypercubes, butterfly-derivative networks, $k$-regular $k$-connected graphs and so on, their $k$-diameters have been studied in Refs.[2–4]. In this paper we discuss the $k$-diameters of some circulant graphs.

Given a graph $G$. For $x, y \in V(G), x \neq y$, let $P_k(x, y)$ be a family of $k$ internally disjoint paths between $x$ and $y$, i.e.

$$P_k(x, y)=\{p_1, p_2, \ldots, p_k\}$$

where $|p_i| \leq |p_2| \leq \cdots \leq |p_k|$ and $|p_i|$ denotes length of the path $p_i, i=1,2,\ldots,k$. The $k$-distance $d_k(x,y)$ between vertices $x$ and $y$ is the minimum $|P_k|$ among all $P_k(x, y)$ and $k$-diameter $d_k(G)$ of $G$ is defined as the maximum $k$-distance $d_k(x, y)$ over all pairs $x, y$ of vertices of $G$.

If a graph $G$ is $k$-connected, by Menger’s Theorem, any pair of two distinct nodes $x$ and $y$ have a $P_k(x, y)$. This means that the $d_k(G)$ is existent if $G$ is $k$-connected.

Lemma 1 [3] If $G$ is a 3-regular 3-connected graph with $2n$ vertices then $d_3(G) \leq n$.

Let $Z_n = \{0,1,\cdots,n-1\}$, $S \subseteq Z_n-\{0\}$, $-S=S$ (mod $n$), namely there exist $j_1, j_2,\cdots, j_r$, such that $S=\{j_1, j_2,\cdots, j_r, n-j_1, n-j_2,\cdots, n-j_r\}$, $j_1, j_2,\cdots, j_r$ are called spanning elements.

Definition 1 The graph $G$ with order $n$ is called circulant graph if it satisfies:

1) $V(G)=Z_n$;
2) $E(G) = \{ij \mid j-i \in S\}$, where the operation takes module $n$.

The graph $G$ in Definition 1 is denoted by $C_n(j_1,$
Let gcd \((x, y)\) be the maximum common divisor of \(x\) and \(y\). It is proved that a circulant graph \(C_n(j_1, j_2, \ldots, j_r)\) is a connected graph if \(gcd (n, j_1, j_2, \ldots, j_r) = 1\).

A connected circulant graph with order \(n (n \geq 3)\) and degree 2 is a \(n\)-cycle \(C_n\). Since \(d_i(C_n) = \frac{n}{2}\),
\[
d_2(C_n) = n - 1, \quad \text{so} \quad d_i(C_n(j)) = \frac{n}{2}, \quad d_2(C_n(j)) = n - 1, \quad \text{where gcd}(n, j) = 1 \quad \text{and}
\]
\[
\left\lfloor \frac{n}{2} \right\rfloor \quad \text{if} \quad n \quad \text{is even}
\]
\[
\left\lfloor \frac{n - 1}{2} \right\rfloor \quad \text{if} \quad n \quad \text{is odd}
\]

By the definition of circulant graph, it is easily seen that Lemma 2 is true.

**Lemma 2** Let \(G\) be a circulant graph of order \(n\) and \(k\) be a positive integer. If \(d_k(G)\) exists, then
\[
d_k(G) = d_k(0, u)
\]
where \(u \in \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}; \) or
\[
d_k(G) = \max_{x \in A} \{d_k(0, x)\}
\]
where \(A = \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \).

**2 Main Result**

**Lemma 3** If \(G = C_n(1, \frac{n}{2})\), \(n \geq 4\) and \(n\) is even, then
1) \(d_1(G) = \left\lfloor \frac{n - 2}{4} \right\rfloor + 1; \) 2) \(d_2(G) = \left\lfloor \frac{n - 2}{4} \right\rfloor + 2; \) 3) \(d_3(G) = \frac{n}{2}\).

**Proof** When \(n = 4\), \(G \equiv K_4\). We know \(d_1(K_4) = 1, d_2(K_4) = 2\) and \(d_3(K_4) = 2\); so \(G\) satisfied Eqs.(1)–(3) if \(n = 4\). We suppose that \(n > 4\) in the following proof.

1) Let \(G' = G - 0\), as Fig.1 shows. We write \(d_k(G; x, y)\) for \(k\)-distance between \(x\) and \(y\) in \(G\). Clearly for \(x \in \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1\}\), we have
\[
d(G; 0, x) = \min \{d(G^*; 1, x), d(G^*; \frac{n}{2}, x), d(G^*; n - 1, x)\} + 1
\]

Let \(G'' = G - \{\frac{n}{2} - 1, \frac{n}{2} - 1\}\), as Fig.2 shows. In \(G'\), vertexes \(\frac{n}{2} - 1\) is adjacent to \(n - 1\) and \(\frac{n}{2}\).

By Eqs.(1)–(3), we get
\[
d(G; 0, x) = \min \{d(G^*; 1, x), d(G^*; \frac{n}{2}, x), d(G^*; n - 1, x)\} + 1
\]

thus, we get
\[
\max_{0 \leq r \leq \frac{n-2}{4}} d(G; 0, x) = \left\lfloor \frac{n-2}{4} \right\rfloor + 1
\]

so

\[
\max_{0 \leq r \leq \frac{n-2}{4}} d(G; 0, x) = \left\lfloor \frac{n-2}{4} \right\rfloor + 1
\]

since \(d(G; 0, \frac{n}{2}) = 1\). By Lemma 2, \(d_1(G) = \left\lfloor \frac{n-2}{4} \right\rfloor + 1\).

2) The vertices which are adjacent to 0 in \(G\) are 1, \(\frac{n}{2}\) and \(n-1\). Let \(G_1 = G - 1\), \(G_2 = G - \frac{n}{2}\), and \(G_3 = G - (n-1)\).

By definition of 2-disdance, for \(x \neq 0\)

\[
d_2(G; 0, x) = \min\{d_2(G; 0, 0), d_2(G; 0, x)\}
\]

(5)

Let \(y = \begin{cases} \frac{n}{4} & n = 4, 8, \ldots \\ \frac{n-2}{4} & n = 6, 10, \ldots \end{cases}\)

It is easily seen from Fig.3 that

\[
d_2(G_2; 0, y) = \max \{|p|, |q|\} = \max \{y, \frac{n}{2} - y + 1\} = \frac{n}{2} - y + 1
\]

(6)

where \(p = 01 \cdots (y-1)y\) and \(q = 0(n-1)(\frac{n}{2} - 1)\cdots (y+1)y\).

Similarly as shown in Fig.4,

\[
d_2(G_3; 0, y) = \max \{y, \frac{n}{2} - y + 1\} = \frac{n}{2} - y + 1
\]

(7)

In \(G_1\) (see Fig.6), let \(r = \frac{n}{2}\) and \(r_2 = 0(n-1)(\frac{n}{2} - 2)\cdots (\frac{n}{2} + y)\).

We have \(|r_1| = |r_2| = \frac{n}{2} - y + 1\) and \(|r_3| = y + 2\). So \(|r_3| \geq |r_2|\).

Thus max \{\(|r_1|, |r_2|\)\} \leq \max \{\{|r_1|, |r_3|\}\}.

\[
d_2(G_1; 0, y) = \max \{|r_1|, |r_2|\} = \frac{n}{2} - y + 1
\]

(8)

since \(r_1\) and \(r_2\) are two internally disjoint paths between 0 and \(y\).

By Eqs.(5)~(8), we get

\[
d_2(0, y) = \max \{d_2(G; 0, 0), d_2(G; 0, x)\} = \left\lfloor \frac{n-2}{4} \right\rfloor + 2
\]

We choose \(x \in \{1, 5, \ldots, \frac{n}{2}\}\), \(x \neq y\).

1) \(x < y\). Let \(l_1 = 012 \cdots x\) and \(l_2 = 0(n-1)(\frac{n}{2} - 1)\cdots (\frac{n}{2} + y)x\). Clearly, \(l_1\) and \(l_2\) are two internally disjoint 0-\(x\) paths (i.e. the path from 0 to \(x\)), and \(\max \{|l_1|, |l_2|\} = |l_2| = x + 1 \leq y + 1 \leq d_2(0, y)\). So \(d_2(0, x) \leq d_2(0, y)\).

2) \(x > y\). Let \(l_3 = 0 \frac{n}{2} (\frac{n}{2} - 1) \cdots x\) and \(l_4 = 0(n-1)(n-2)\cdots (\frac{n}{2} + x)y\). Clearly, \(l_3\) and \(l_4\) are two internally disjoint 0-\(x\) paths, and \(|l_3| = |l_4| = \frac{n}{2} - x + 1 < \frac{n}{2} - y + 1 = d_2(0, y)\). So \(d_2(0, x) \leq |l_3| = d_2(0, y)\).

By Lemma 2, we get

\[
d_2(G) = d_2(0, y) = \left\lfloor \frac{n-2}{4} \right\rfloor + 2
\]

3) Let \(p_1^* = 01, p_2^* = 0(n-1)(\frac{n}{2} + 1)\) and \(p_3^* = 0(n-1)(\frac{n}{2} - 1)\cdots 21\). By Fig.1, it is easily known that:

1) \(p_1^*\), \(p_2^*\) and \(p_3^*\) are three internally disjoint 0-1 paths; (2) \(p_1^*\) is the shortest 0-1 path among all 0-1 paths and \(|p_1^*| = 1\); (3) \(|p_2^*| = 3 = \min \{|p|\}\), where \(T_2 = \)
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\( \{p \mid p \text{ is a 0-1 path and } \frac{n}{2} \in V(p)\} \); 

(4) \( |p_3^*| = \frac{n}{2} = \min \{ |p| \} \geq 3 \), where \( I_3 = \{ p \mid p \text{ is a 0-1 path} \}, n-1 \in V(p) \) and \( \frac{n}{2} \notin V(p) \).

Since the vertex 0 is just adjacent to 1, \( \frac{n}{2} \) and \( n-1 \), for arbitrary three internally disjoint 0-1 paths \( p_1, p_2 \) and \( p_3 \), it is easily known that \( p_1 = p_1^*, p_2 \in I_2 \) and \( p_3 \in I_3 \). So

\[
\begin{align*}
&d_1(0,1) = \min_{p_1, p_2, p_3 \text{ are three internally disjoint 0-1 paths}} \max \{|p_1|, |p_2|, |p_3|\} \\
&\quad = \max \{|p_1^*|, |p_2^*|, |p_3^*|\} = \frac{n}{2}
\end{align*}
\]

Thus \( d_3(G) \geq \frac{n}{2} \).

Ref. [1] shows that \( G = C_n(1, \frac{n}{2}) \) is a 3-connected graph. Thus \( G \) is a 3-regular 3-connected on \( n \) vertices.

By Lemma 1, we have \( d_3(G) \leq \frac{n}{2} \) and \( d_3(G) = \frac{n}{2} \).

**Theorem 1** Let circulant graph \( G = C_n(i, \frac{n}{2}) \), where \( n \geq 4 \) and \( n \) is even. If \( \gcd(n, i) = 1 \), then

1) \( d_1(G) = \left\lceil \frac{n-2}{4} \right\rceil + 1; \quad 2) \ d_2(G) = \left\lceil \frac{n-2}{4} \right\rceil + 2; \quad 3) \ d_3(G) = \frac{n}{2} \).

**Proof** For a circulant graph \( C_n(i, \frac{n}{2}) \), Ref. [5] shows that if \( \gcd(n, i) = 1 \) then \( C_n(i, \frac{n}{2}) \cong C_n(1, \frac{n}{2}) \). Thus, by Lemma 3, Eqs. (1)–(3) all are true.

**References**


**Brief Introduction to Author(s)**

ZHANG Xian-di (张先迪) was born in 1947. He is now a professor at UESTC. His research interests include graph theory and its application.