Representation Theorem of Fuzzy Rough Set*

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Abstract This paper gives the definition of $\lambda$-cut sets and studies the structure of fuzzy rough sets. Based on the concept of rough sets, this paper proposes the representation theorem of fuzzy rough sets.

Key words rough set; fuzzy sets; fuzzy rough sets; $\lambda$-cut sets; representation theorem

Rough set theory is emerging as a powerful tool for reasoning about data. It can be used in some branches of artificial intelligence. Fuzzy rough sets are studied on bases of fuzzy sets and rough sets[1]. Fuzzy sets and rough sets have been received much attention in recent years[2]. In contrast, the research about fuzzy rough sets is in its developing stage. The purpose of this paper is to introduce and discuss the representation theorems of fuzzy rough sets to show the structure of fuzzy rough sets.

1 Preliminaries and Basic Definitions

In this section, the basis of our theorems is build. We start the discussion firstly from the concepts of fuzzy sets and rough sets. Then the definitions of $\lambda$-cut sets and nested sets are introduced.

Definition 1 Let $U$ be a nonempty set. Let $\langle L, \leq \rangle$ be a lattice and $\&$ be a Boolean subalgebra of the set of all subsets of $U$. Now consider a rough set $X = (X_L, X_U) \in \&^2$ with $X_L \subseteq X_U$. A fuzzy rough set in $X$ is an object of the form $A = (A_L, A_U)$, where $A_L$ and $A_U$ are characterized by a pair of maps $\mu_{A_L}: X_L \rightarrow L$, $\mu_{A_U}: X_U \rightarrow L$ with the property $\mu_{A_L}(x) \leq \mu_{A_U}(x)$ for all $x \in X_L$.

Definition 2 Let $A_L: X_L \rightarrow L$ (note $A_L \in L^{X_L}$) $A_U: X_U \rightarrow L$ (note $A_U \in L^{X_U}$), $(X_L, X_U)$ is a rough set, $L$ is lattice. For $\forall \lambda \in L$, we define that

$A_{L_{\lambda}} = \{x \in X_L | A_L(x) < \lambda\} \subseteq X_L$ 

$A_{U_{\lambda}} = \{x \in X_U | A_U(x) < \lambda\} \subseteq X_U$

We call $(A_{L_{\lambda}}, A_{U_{\lambda}})$ a $\lambda$- lower cut set and $(A_{L_{\lambda}}, A_{U_{\lambda}})$ a $\lambda$- strong lower cut set on $(A_{L_{\lambda}}, A_{U_{\lambda}})$ of fuzzy rough sets.

Similarly, we define that

$A_{L_{1-\lambda}} = \{x \in X_L | A_L(x) > 1 - \lambda\} \subseteq X_L$ 

$A_{U_{1-\lambda}} = \{x \in X_U | A_U(x) > 1 - \lambda\} \subseteq X_U$

We call $(A_{L_{1-\lambda}}, A_{U_{1-\lambda}})$ a $\lambda$- lower quasi-cut set and $(A_{L_{1-\lambda}}, A_{U_{1-\lambda}})$ a $\lambda$- strong lower quasi-cut set on $(A_{L_{\lambda}}, A_{U_{\lambda}})$ of fuzzy rough sets.

And we further define that

$A_{L_{\lambda}} = \{x \in X_L | A_L(x) \leq 1 - \lambda\} \subseteq X_L$ 

$A_{U_{\lambda}} = \{x \in X_U | A_U(x) \leq 1 - \lambda\} \subseteq X_U$

We call $(A_{L_{\lambda}}, A_{U_{\lambda}})$ a $\lambda$- upper quasi-cut set and $(A_{L_{\lambda}}, A_{U_{\lambda}})$ a $\lambda$- strong upper quasi-cut set on $(A_{L_{\lambda}}, A_{U_{\lambda}})$ of fuzzy rough sets.

Definition 3 Let $R$ be a family of rough sets and $L$ be a complete lattice, $H = (H_L, H_U)$, $(H_L: L \rightarrow P(R_L); H_U: L \rightarrow P(R_U))$ is said to be an antitone nested set if
Let $H_L(\mu) \subseteq H_U(\lambda)$ iff \( \lambda \in L, \mu \in L, \) and \( \lambda < \mu \).

\[ \bigcap_{\alpha \in \mathcal{L}} H_L(\lambda) \subseteq \bigcap_{\alpha \in \mathcal{L}} H_U(\alpha) \]

iff \( \{ \lambda, \forall t \in T \} \subseteq L \) and \( \lambda = \bigvee_{t \in T} \lambda_t \).

**Definition 4** Let $HL(R)$ be sets of antitonic nested sets of $R$, we define

\[
\cup_{\text{def}} H_L : (\cup_{\text{def}} H_L)(\lambda) = \cup_{\text{def}} H_L(\lambda) \\
\cap_{\text{def}} H_L : (\cap_{\text{def}} H_L)(\lambda) = \cap_{\text{def}} H_L(\lambda) \\
\cup_{\text{def}} H_U : (\cup_{\text{def}} H_U)(\lambda) = \cup_{\text{def}} H_U(\lambda) \\
\cap_{\text{def}} H_U : (\cap_{\text{def}} H_U)(\lambda) = \cap_{\text{def}} H_U(\lambda) \\
H^L : H^L(\lambda) = (H_L(1-\lambda))^c \\
H^U : H^U(\lambda) = (H_U(1-\lambda))^c
\]

Then $(HL(R), \cup, \cap, c)$ be majorized soft algebra.

Similarly, we define

\[
\cup_{\text{def}} H_L : (\cup_{\text{def}} H_L)(\lambda) = \cup_{\text{def}} H_L(\lambda) \\
\cap_{\text{def}} H_L : (\cap_{\text{def}} H_L)(\lambda) = \cap_{\text{def}} H_L(\lambda) \\
\cup_{\text{def}} H_U : (\cup_{\text{def}} H_U)(\lambda) = \cup_{\text{def}} H_U(\lambda) \\
\cap_{\text{def}} H_U : (\cap_{\text{def}} H_U)(\lambda) = \cap_{\text{def}} H_U(\lambda) \\
H^L : H^L(\lambda) = (H_L(1-\lambda))^c \\
H^U : H^U(\lambda) = (H_U(1-\lambda))^c
\]

**Definition 5** Let $R$ be a family of rough sets and $L$ be a complete lattice, $H=(H_L,H_U)$, $(H_L : L \rightarrow P(R_L); H_U : L \rightarrow P(R_U))$ is said to be a sequential nested set if

\[ H_L(\lambda) \subseteq H_U(\mu) \]

iff \( \lambda \in L, \mu \in L, \) and \( \lambda < \mu \).

\[ \bigcap_{\alpha \in \mathcal{L}} H_L(\lambda) \subseteq \bigcap_{\alpha \in \mathcal{L}} H_U(\alpha) \]

iff \( \{ \lambda, \forall t \in T \} \subseteq L \) and \( \lambda = \bigvee_{t \in T} \lambda_t \).

2 Representation Theorems of Fuzzy Rough Sets

In this section, we give the representation theorems of fuzzy rough sets to show the structure of fuzzy rough sets.

**Theorem 1** Let $L$ be dense complete lattice and $T : HL(R) \rightarrow L^\lambda$, $T(H) = \bigcap_{\lambda \in \mathcal{L}} \lambda H(\lambda)$, where $T$ is an epimorphism, or say, subjection, and

\[ T(H_L) = \bigcap_{\lambda \in \mathcal{L}} \lambda H_L(\lambda) \quad T(H_U) = \bigcap_{\lambda \in \mathcal{L}} \lambda H_U(\lambda) \]

Then, we have

\[ T(H_L)^{[1]} \subseteq H_L(\lambda) \subseteq T(H_L)^{[1]} \]

\[ T(H_U)^{[1]} \subseteq H_U(\lambda) \subseteq T(H_U)^{[1]} \]

\[ T(H_L)^{[1]} = \bigcap_{\lambda \in \mathcal{L}} H_L(\alpha) \quad T(H_U)^{[1]} = \bigcap_{\lambda \in \mathcal{L}} H_U(\alpha) \]

and

\[ T(\cup_{\text{def}} H_U) = \bigcap_{\lambda \in \mathcal{L}} T(H_U) \quad T(\cap_{\text{def}} H_U) = \bigcap_{\lambda \in \mathcal{L}} T(H_U) \]

\[ T(H^L) = T(H_L)^c \quad T(H^U) = T(H_U)^c \]

**Theorem 2** Let $L$ be dense complete lattice and $T : HL(R) \rightarrow L^\lambda$, $T(H) = \bigcup_{\lambda \in \mathcal{L}} \lambda H(\lambda)$, where $T$ is an epimorphism, or say, subjection, and

\[ T(H_L) = \bigcup_{\lambda \in \mathcal{L}} \lambda H_L(\lambda) \quad T(H_U) = \bigcup_{\lambda \in \mathcal{L}} \lambda H_U(\lambda) \]

Then, we have

\[ T(H_L)^{[1]} \subseteq H_L(\lambda) \subseteq T(H_L)^{[1]} \]

\[ T(H_U)^{[1]} \subseteq H_U(\lambda) \subseteq T(H_U)^{[1]} \]

\[ T(H_L)^{[1]} = \bigcup_{\lambda \in \mathcal{L}} H_L(\alpha) \quad T(H_U)^{[1]} = \bigcup_{\lambda \in \mathcal{L}} H_U(\alpha) \]

and

\[ T(\cup_{\text{def}} H_U) = \bigcup_{\lambda \in \mathcal{L}} T(H_U) \quad T(\cap_{\text{def}} H_U) = \bigcup_{\lambda \in \mathcal{L}} T(H_U) \]

\[ T(H^L) = T(H_L)^c \quad T(H^U) = T(H_U)^c \]

**Theorem 3** Let $L$ be dense complete lattice and $T : HL(R) \rightarrow L^\lambda$, $T(H) = \bigcap_{\lambda \in \mathcal{L}} \lambda H(\lambda)$, where $T$ is an epimorphism, or say, subjection, and

\[ T(H_L) = \bigcap_{\lambda \in \mathcal{L}} \lambda H_L(\lambda) \quad T(H_U) = \bigcap_{\lambda \in \mathcal{L}} \lambda H_U(\lambda) \]

Then, we have

\[ T(H_L)^{[1]} \subseteq H_L(\lambda) \subseteq T(H_L)^{[1]} \]
\[ T(H_L^\lambda)^c \subseteq H_L(\lambda) \subseteq T(H_L^\lambda) \]
\[ T(H_L^\alpha)^c = \bigcap_{\alpha \in L} H_L(\alpha) \quad T(H_L^\alpha) = \bigcap_{\alpha \in L} H_L(\alpha) \]
\[ T(H_L^\alpha)^c = \bigcap_{\alpha \in L} H_L(\alpha) \quad T(H_L^\alpha) = \bigcap_{\alpha \in L} H_L(\alpha) \]

and
\[ T(\bigcup_{\gamma \in \Gamma} H_L^\gamma) = \bigcup_{\gamma \in \Gamma} T(H_L^\gamma) \]
\[ T(\bigcap_{\gamma \in \Gamma} H_L^\gamma) = \bigcap_{\gamma \in \Gamma} T(H_L^\gamma) \]
\[ T(\bigcap_{\gamma \in \Gamma} H_L^\gamma) = \bigcap_{\gamma \in \Gamma} T(H_L^\gamma) \]
\[ T(H_L^\gamma) = T(H_L^\gamma)^c \quad T(H_L^\gamma)^c = T(H_L^\gamma) \]

**Proof of Theorems**

Starting from the concepts of fuzzy sets and rough sets, we describe the definitions of \( \lambda \)-cut sets and study the structure of fuzzy rough sets in Section 1. According to these definitions, the representation theorems of fuzzy rough sets are proposed above.

Now we prove Theorem 1. The proofs of the other two theorems can be done by a similar procedure.

For Eq. (1), \( \exists \alpha \in L \), we have
\[ \forall x \in T(H_L^\lambda) \quad T(H_L^\lambda)(x) < 1 - \lambda \Rightarrow \]
\[ \land_{\alpha \in L} (\alpha \land H_L(\alpha))(x) < 1 - \lambda \Rightarrow \]
\[ \alpha \land H_L(\alpha)(x) < 1 - \lambda \Rightarrow 1 - \alpha < 1 - \lambda \]

and \( x \in H_L(\alpha) \), \( H_L(\alpha) \subseteq H_L(\lambda) \), so \( x \in H_L(\lambda) \).
Therefore, we obtain
\[ T(H_L^\lambda) \subseteq H_L(\lambda) \]

And
\[ \forall x \in H_L(\lambda) \Rightarrow H_L(\lambda)(x) = 1 \]

Thus \( x \in T(H_L^\lambda) \), that is \( H_L(\lambda) \subseteq T(H_L^\lambda) \). We achieve that
\[ T(H_L^\lambda) \subseteq H_L(\lambda) \subseteq T(H_L^\lambda) \]

Similarly, we can prove
\[ T(H_L^\lambda) \subseteq H_L(\lambda) \subseteq T(H_L^\lambda) \]

**3 Conclusions**

Rough set theory and fuzzy set theory have strong supplement. So fuzzy rough sets are studied on bases of fuzzy sets and rough sets. This paper gives the representation theorems of fuzzy rough sets to show the structure of fuzzy rough sets.

**References**


**Brief Introduction to Author(s)**

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