Theoretical Proof of Unconditional Stability of the 3-D ADI-FDTD Method*

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Abstract In order to eliminate Courant-Friedrich-Levy (CFL) condition restraint and improve computational efficiency, a new finite-difference time-domain (FDTD) method based on the alternating-direction implicit (ADI) technique is introduced recently. In this paper, a theoretical proof of the stability of the three-dimensional (3-D) ADI-FDTD method is presented. It is shown that the 3-D ADI-FDTD method is unconditionally stable and free from the CFL condition restraint.

Key words alternating-direction implicit (ADI) technique; Courant-Friedrich-Levy (CFL) condition restraint; finite-difference time-domain (FDTD) method; stability

The finite-difference time-domain (FDTD) method is a very useful numerical simulation technique for solving electromagnetic problem[1]. Because the traditional FDTD method is based on an explicit finite-difference algorithm, the maximum time step must be small enough so as to satisfy the Courant-Friedrich-Levy (CFL) stability condition[2]. Therefore, it will cause excessive computation memory and time for problems with electrically small local structures, and largely limit FDTD applications for electromagnetic simulation.

Recently, a new FDTD algorithm free from the CFL stability condition has been proposed[3~5]. This new algorithm is based on the alternating direction implicit (ADI) technique and is applied to Yee’s staggered cell to solve Maxwell’s equations. In Ref. [3], the ADI-FDTD formulation for the two-dimensional TE wave case has been given and shown to be unconditionally stable. Consequently, when the ADI-FDTD method is used, the limitation of the maximum time-step size of the method does not depend on the CFL stability condition, but rather on the numerical dispersion.

Applications of the ADI-FDTD method to 3-D problems have been given in Refs.[4,5] without theoretical proof of the unconditional stability. A theoretical proof of unconditional stability is introduced from the view of the growth matrix in spatial spectral domain[6]. In this paper, details of another simpler theoretical proof of the unconditional stability for the 3-D ADI-FDTD method are presented.

1 Difference Scheme of the 3-D ADI-FDTD Method

In an isotropic, time-invariant, lossless and source-free medium, Maxwell’s curl equations can be represented by the following six coupled scalar equations in Cartesian coordinates:

\[
\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} \right) \quad (1)
\]
\[
\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) \quad (2)
\]
\[
\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \right) \quad (3)
\]
\[
\frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \quad (4)
\]
\[
\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \quad (5)
\]
\[
\frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) \quad (6)
\]

The numerical formulation of the 3-D ADI-FDTD
method is presented in Eq. (7)~(18). It consists of two procedures. In the 3-D ADI-FDTD algorithm, we still adopt the Yee's staggered cell to solve Maxwell's equations. In procedure one, the first term of the right-hand side is replaced by an implicit difference approximation, while the second term by an explicit difference approximation. In procedure two, the second term of the right-hand side is replaced by an implicit difference approximation, while the first term by an explicit difference approximation.

Procedure one

\[
E_{x}^{n+1}(i+\frac{1}{2}, j, k) - E_{x}^{n}(i+\frac{1}{2}, j, k) = \frac{H_{y}^{n+1}(i+\frac{1}{2}, j+\frac{1}{2}, k) - H_{y}^{n+1}(i+\frac{1}{2}, j-\frac{1}{2}, k)}{\Delta t} + \frac{E_{x}^{n}(i+\frac{1}{2}, j, k) - E_{x}^{n}(i+\frac{1}{2}, j+1, k)}{\Delta z} \tag{7}
\]

\[
H_{y}^{n+1}(i+\frac{1}{2}, j, k+\frac{1}{2}) - H_{y}^{n}(i+\frac{1}{2}, j, k+\frac{1}{2}) = \frac{E_{y}^{n+1}(i+\frac{1}{2}, j, k+\frac{1}{2}) - E_{y}^{n+1}(i+\frac{1}{2}, j, k+\frac{1}{2})}{\Delta t} \tag{10}
\]

\[
E_{y}^{n+1}(i+1, j, k+\frac{1}{2}) - E_{y}^{n}(i+1, j, k+\frac{1}{2}) = \frac{E_{y}^{n}(i+1, j, k) - E_{y}^{n}(i+1, j, k+1)}{\Delta y} \tag{11}
\]

Procedure two

\[
E_{y}^{n+1}(i+1, j+\frac{1}{2}, k) - E_{y}^{n}(i+1, j+\frac{1}{2}, k) = \frac{H_{x}^{n+1}(i+\frac{1}{2}, j+\frac{1}{2}, k) - H_{x}^{n+1}(i+\frac{1}{2}, j, k-\frac{1}{2})}{\Delta t} \tag{12}
\]

\[
H_{x}^{n+1}(i+\frac{1}{2}, j, k-\frac{1}{2}) - H_{x}^{n+1}(i+\frac{1}{2}, j, k+\frac{1}{2}) = \frac{E_{y}^{n+1}(i+\frac{1}{2}, j, k-\frac{1}{2}) - E_{y}^{n+1}(i+\frac{1}{2}, j, k+\frac{1}{2})}{\Delta z} \tag{13}
\]

\[
E_{y}^{n+1}(i+1, j+1, k) - E_{y}^{n}(i+1, j+\frac{1}{2}, k) = \frac{H_{x}^{n+1}(i+\frac{1}{2}, j, k+\frac{1}{2}) - H_{x}^{n+1}(i+\frac{1}{2}, j-\frac{1}{2}, k)}{\Delta t} \tag{14}
\]

\[
H_{x}^{n+1}(i+\frac{1}{2}, j-\frac{1}{2}, k) - H_{x}^{n+1}(i+\frac{1}{2}, j+\frac{1}{2}, k) = \frac{E_{y}^{n+1}(i+\frac{1}{2}, j-\frac{1}{2}, k) - E_{y}^{n+1}(i+\frac{1}{2}, j+\frac{1}{2}, k)}{\Delta z} \tag{15}
\]
\[
H_{x}^{n+1}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) - H_{x}^{n}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) = \Delta t \frac{\Delta x}{\mu} \left[ E_{x}^{n+1}\left(i,j,k+1\right) - E_{x}^{n+1}\left(i,j,k\right) \right] + \Delta t \frac{\Delta y}{\varepsilon} \left[ E_{y}^{n+1}\left(i,j+1,k\right) - E_{y}^{n+1}\left(i,j,k\right) \right] + \Delta t \frac{\Delta z}{\mu} \left[ E_{z}^{n+1}\left(i,j,k+1\right) - E_{z}^{n+1}\left(i,j,k\right) \right]
\]

\[
H_{y}^{n+1}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) - H_{y}^{n}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) = \Delta t \frac{\Delta x}{\mu} \left[ E_{x}^{n+1}\left(i+1,j,k+\frac{1}{2}\right) - E_{x}^{n+1}\left(i,j,k+\frac{1}{2}\right) \right] + \Delta t \frac{\Delta y}{\varepsilon} \left[ E_{y}^{n+1}\left(i,j+1,k\right) - E_{y}^{n+1}\left(i,j,k\right) \right] + \Delta t \frac{\Delta z}{\mu} \left[ E_{z}^{n+1}\left(i,j,k+1\right) - E_{z}^{n+1}\left(i,j,k\right) \right]
\]

\[
H_{z}^{n+1}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) - H_{z}^{n}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) = \Delta t \frac{\Delta x}{\mu} \left[ E_{x}^{n+1}\left(i+1,j,k+\frac{1}{2}\right) - E_{x}^{n+1}\left(i,j,k+\frac{1}{2}\right) \right] + \Delta t \frac{\Delta y}{\varepsilon} \left[ E_{y}^{n+1}\left(i,j+1,k\right) - E_{y}^{n+1}\left(i,j,k\right) \right] + \Delta t \frac{\Delta z}{\mu} \left[ E_{z}^{n+1}\left(i,j,k+1\right) - E_{z}^{n+1}\left(i,j,k\right) \right]
\]

2 Numerical Stability

The numerical stability of the combination of procedure one and procedure two can be analytically derived by using the Von Neumann method. The plane wave eigenmodes of a 3-D electromagnetic wave are defined as follows:

\[
E_{x}^{n}(i,j,k) = E_{0x} e^{\xi_{x} n} \exp\left[i(k_{x} \Delta x + jk_{y} \Delta y + kk_{z} \Delta z)\right]
\]

\[
E_{y}^{n}(i,j,k) = E_{0y} e^{\xi_{y} n} \exp\left[i(k_{x} \Delta x + jk_{y} \Delta y + kk_{z} \Delta z)\right]
\]

\[
E_{z}^{n}(i,j,k) = E_{0z} e^{\xi_{z} n} \exp\left[i(k_{x} \Delta x + jk_{y} \Delta y + kk_{z} \Delta z)\right]
\]

where \( \xi_{x} = \sqrt{-1}, k_{x}, k_{y}, \) and \( k_{z} \) are the x-, y- and z-components of numerical wavevector \( k \), respectively. \( \xi \) indicates the growth factor of each component at each time step. Substituting the eigenmode expressions of Eq.(19) into Eqs.(7)~(12) in procedure one, we obtain

\[
\begin{bmatrix}
\bar{\xi}_{1} - 1 & 0 & 0 & 0 & \bar{\xi}_{1} q_{x} & -\bar{\xi}_{1} q_{y} & 0 \\
0 & \bar{\xi}_{1} - 1 & 0 & -\bar{\xi}_{1} q_{y} & 0 & \bar{\xi}_{1} q_{x} & 0 \\
0 & 0 & \bar{\xi}_{1} - 1 & \bar{\xi}_{1} q_{x} & -\bar{\xi}_{1} q_{y} & 0 & 0 \\
0 & -\bar{\xi}_{1} q_{x} & 0 & \bar{\xi}_{1} - 1 & 0 & \bar{\xi}_{1} q_{y} & 0 \\
0 & \bar{\xi}_{1} q_{y} & 0 & 0 & \bar{\xi}_{1} - 1 & 0 & \bar{\xi}_{1} q_{x} \\
0 & -\bar{\xi}_{1} q_{y} & 0 & 0 & 0 & \bar{\xi}_{1} - 1 & 0 \\
\end{bmatrix}
\times
\begin{bmatrix}
E_{0x} \\
E_{0y} \\
E_{0z} \\
H_{0x} \\
H_{0y} \\
H_{0z}
\end{bmatrix}
= 0
\]

To guarantee the existence of solutions to the linear homogeneous system, the determinant to the

\[
q_{x} = \frac{2 \Delta t}{\varepsilon \Delta x} \sin\left(\frac{k_{x} \Delta x}{2}\right) \\
q_{y} = \frac{2 \Delta t}{\varepsilon \Delta y} \sin\left(\frac{k_{y} \Delta y}{2}\right) \\
q_{z} = \frac{2 \Delta t}{\mu \Delta z} \sin\left(\frac{k_{z} \Delta z}{2}\right)
\]
coefficient matrix must be zero. This result is
\[
\begin{bmatrix}
\xi_i^2 P_{x}\xi_i^2 P_{y} - M_i(\xi_i) L_i(\xi_i) \\
\xi_i^2 P_{x}\xi_i^2 P_{y} + M_i(\xi_i) N_i(\xi_i)
\end{bmatrix}
\begin{bmatrix}
\xi_i^2 P_{x}\xi_i^2 P_{y} - M_i(\xi_i) L_i(\xi_i) \\
\xi_i^2 P_{x}\xi_i^2 P_{y} + M_i(\xi_i) N_i(\xi_i)
\end{bmatrix}
\begin{bmatrix}
P_{x}\xi_i^2 P_{y} + M_i(\xi_i) P_{x}\xi_2 \\
P_{x}\xi_i^2 P_{y} + M_i(\xi_i) P_{x}\xi_2
\end{bmatrix}
\]
(21)

where
\[
P_i = q_i r_i, \quad P_y = q_i r_i, \quad P_z = q_i r_i,
\]
\[
M_i(\xi_i) = (\xi_i - 1)^2 + \xi_i^2 P_x + P_z,
\]
\[
N_i(\xi_i) = (\xi_i - 1)^2 + \xi_i^2 P_x + P_z,
\]
\[
L_i(\xi_i) = (\xi_i - 1)^2 + \xi_i^2 P_x + P_z.
\]

Similarly, we can derive the equation satisfied by

\[
\begin{bmatrix}
\xi_i^2 P_{x}\xi_i^2 P_{y} - M_i(\xi_i) L_i(\xi_i) \\
\xi_i^2 P_{x}\xi_i^2 P_{y} + M_i(\xi_i) N_i(\xi_i)
\end{bmatrix}
\begin{bmatrix}
\xi_i^2 P_{x}\xi_i^2 P_{y} - M_i(\xi_i) L_i(\xi_i) \\
\xi_i^2 P_{x}\xi_i^2 P_{y} + M_i(\xi_i) N_i(\xi_i)
\end{bmatrix}
\begin{bmatrix}
P_{x}\xi_i^2 P_{y} + M_i(\xi_i) P_{x}\xi_2 \\
P_{x}\xi_i^2 P_{y} + M_i(\xi_i) P_{x}\xi_2
\end{bmatrix}
\]
(22)

where
\[
M_i(\xi_i) = (\xi_i - 1)^2 + \xi_i^2 P_x + P_z,
\]
\[
N_i(\xi_i) = (\xi_i - 1)^2 + \xi_i^2 P_x + P_z,
\]
\[
L_i(\xi_i) = (\xi_i - 1)^2 + \xi_i^2 P_x + P_z.
\]

Let \( \xi_i = \frac{1}{\beta} \) and substitute it for Eq.(21), we obtain
\[
\begin{bmatrix}
P_{x}\xi_i^2 P_{y} - \beta^2 M_i(\frac{1}{\beta}) L_i(\frac{1}{\beta}) \\
P_{x}\xi_i^2 P_{y} + \beta^2 M_i(\frac{1}{\beta}) P_{x}\xi_i^2 P_{y} + \beta M_i(\frac{1}{\beta}) P_{x}\xi_i^2 P_{y} + \beta M_i(\frac{1}{\beta})
\end{bmatrix}
\begin{bmatrix}
P_{x}\xi_i^2 P_{y} - \beta^2 M_i(\frac{1}{\beta}) L_i(\frac{1}{\beta}) \\
P_{x}\xi_i^2 P_{y} + \beta M_i(\frac{1}{\beta}) P_{x}\xi_i^2 P_{y} + \beta M_i(\frac{1}{\beta})
\end{bmatrix}
\]
(23)

since
\[
M_i(\frac{1}{\beta}) = (\frac{1}{\beta} - 1)^2 + \frac{\beta^2}{\beta^2} P_x + P_z = \frac{M_i(\beta)}{\beta^2},
\]
\[
N_i(\frac{1}{\beta}) = (\frac{1}{\beta} - 1)^2 + \frac{\beta^2}{\beta^2} P_x + P_z = \frac{N_i(\beta)}{\beta^2},
\]
\[
L_i(\frac{1}{\beta}) = (\frac{1}{\beta} - 1)^2 + \frac{\beta^2}{\beta^2} P_x + P_z = \frac{L_i(\beta)}{\beta^2}.
\]

Eq.(21) can be finally transformed into
\[
\begin{bmatrix}
\beta^2 P_{x}\beta^2 P_{y} - M_z(\beta) L_z(\beta) \\
\beta^2 P_{x}\beta^2 P_{y} + M_z(\beta) N_z(\beta)
\end{bmatrix}
\begin{bmatrix}
\beta^2 P_{x}\beta^2 P_{y} - M_z(\beta) L_z(\beta) \\
\beta^2 P_{x}\beta^2 P_{y} + M_z(\beta) N_z(\beta)
\end{bmatrix}
\begin{bmatrix}
\xi_i^2 P_{x}\xi_i^2 P_{y} + M_z(\beta) P_{x}\xi_i^2 P_{y} + M_z(\beta) P_{x}\xi_i^2 P_{y} + M_z(\beta)
\beta^2 P_{x}\beta^2 P_{y} + M_z(\beta) P_{x}\beta^2 P_{x}\beta^2 P_{y} + M_z(\beta) P_{x}\beta^2 P_{x}\beta^2 P_{y} + M_z(\beta)
\end{bmatrix}
\]
(24)

Comparing Eq.(24) with Eq.(22), we find that both equations are the same except for the unknown variables \( \xi_2 \) and \( \beta \). Therefore, the growth factor \( \xi_2 \) must be equal to \( \beta \). Finally, the total growth factor of procedure one plus procedure two is
\[
\xi = \sqrt{\xi_1^2 \xi_2^2} = \xi_1^2 \xi_2^2 = \frac{1}{\beta} |\beta| = 1
\]
(25)

which is always satisfied so that the 3-D ADI-FDTD algorithm is unconditionally stable in any case.

3 Conclusions

In this paper, we introduce the 3-D ADI-FDTD method for solving electromagnetic problems, and strictly prove its unconditional stability. The proof is much simpler than that given in Ref.[6]. Different from the conventional FDTD method, the limitation of the maximum time-step size of ADI-FDTD method only depends on numerical error. For problems needing very small nonuniform local cells in the computational domain, the 3-D ADI-FDTD method is more efficient than the conventional FDTD method.

References


Brief Introduction to Author(s)

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