New Stability Condition of T-S Fuzzy Systems and Design of Robust Flight Control Principle

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Abstract—Unlike the previous research works analyzing the stability of the T-S (Takagi–Sugeno) fuzzy model, an extension on the stability condition of T-S fuzzy systems with a different strategy is provided. In the strategy a new variable, which is relative to the grade of fuzzy membership function, is introduced to the stability analysis and a new stability conclusion is deduced. The definition of stability condition in this paper is different from previous works, though they are similar in form. With the proposed method, the simulation in flight control law shows a better effectiveness.

Index Terms—Flight control, linear matrix inequalities, stability, Takagi-Sugeno (T-S) fuzzy control.

1. Introduction

Comparing with traditional control methods in nonlinear systems, the fuzzy controller has shown better performance that could model uncertainty or nonlinear characteristic[1]. But it is difficult to describe the stability of this nonlinear fuzzy control system. Takagi and Sugeno presented an analytical model of fuzzy system in 1985, called T-S fuzzy model[2]. They analyzed the stability condition and designed the fuzzy controller using the Lyapunov direct method. This T-S model becomes the mostly adopted method in stability analysis of fuzzy control system. The focus of stability analysis about T-S fuzzy system is how to relax the stability conditions and how to find a better controller in an enlarged solution range for the closed-loop system. Tanaka proved a sufficient condition of fuzzy system by seeking a common positive definite matrix \( P \), and the system was asymptotically stable if there existed such matrix[3]. Tanaka also proposed several new relaxed stability conditions and controller design procedures in [3]–[6]. Cao found it was difficult to get a common positive definite matrix \( P_i \) instead of seeking the single common positive definite matrix \( P \)[3]. Yang et al. derived the relaxed stability of the fuzzy control system with uncertain grades of membership which represented the parameter uncertainties of the fuzzy systems[8]. Xie et al. discussed the relaxed stability conditions of continuous T-S fuzzy systems via non-quadratic Lyapunov function technique[9],[10]. They solved the problems (equal to configure the eigenvalues of the closed-loop system) by the stability condition mentioned before, which are all the singly feasible solution. However, sometimes it is not sufficient to all the subsystems, so Chadli et al. studied the stability farther but the conclusion is still not perfect for all subsystems[11]. This paper studies more deeply on the stability research works of [2], [10], and [11]; a different stability condition is proved through Lyapunov stability theory, and a simulation result of flight control system is given comparing with the basic stability conditions in [2].

The paper is organized as follows. Section 2 outlines the T-S fuzzy system model and the basic stability condition by a different analysis method and proves a new stability condition. Section 3 mainly presents the simulation effectiveness of flight control law with the proposed method. Conclusions are given in Section 4.

2. T-S Fuzzy Model and the Different Relaxed Stability Results

2.1 Description of Fuzzy Model

The ordinary T-S continuous fuzzy system is described as

\[
\frac{dx(t)}{dt} = A_r x(t) + B_r u(t)
\]

\[
y(t) = C_r x(t)
\]

\[
u(t) = -F_r x(t), \quad i = 1, 2, \ldots, r
\]
where $M_{i,j}(j=1,2,\cdots,g)$ is the fuzzy sets, $x(t)$ is the state vector, $u(t)$ is the input vector, and $y(t)$ is output vector, $x(t)\in \mathbb{R}^{n_1}$, $y(t)\in \mathbb{R}^{n_2}$, $u(t)\in \mathbb{R}^{n_3}$, $A_i \in \mathbb{R}^{n_1 \times n_1}$, $B_i \in \mathbb{R}^{n_2 \times n_1}$, $C_j \in \mathbb{R}^{n_3 \times n_2}$, and $F_j \in \mathbb{R}^{n_3 \times n_3}$, $r$ is the number of “IF-THEN” rules, $n$ is the number of state variables, $m$ is the number of input variables, $k$ is the number of output variables, and $z_i(t), z_j(t), \cdots, z_q(t)$ are the premise variables.

Given a pair of $(x(t), u(t))$, if we design fuzzy local state feedback controller based on each fuzzy sub-system (such as parallel distributed compensation[6]) and use fuzzy reasoning method and fuzzification process, the finally output of the fuzzy system (1) is given by

$$
\frac{d}{dt}x(t) = \sum_{i=1}^{r} \omega_i(z(t))[A_i x(t) + B_i u(t)] = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t))
$$

$$
y(t) = \sum_{i=1}^{r} \omega_i(z(t))C_i x(t) = \sum_{i=1}^{r} h_i(z(t))C_i x(t). \tag{2}
$$

According to the design method of linear system theory, if ($A_i, B_i$), $j=1,2,\cdots, r$ is a pair of controllable matrices, the fuzzy system is local controllable, then we can assign the eigenvalues of ($A_i, B_i$) at will.

The open-loop system ($u(t)=0$) of (2) is:

$$
\frac{d}{dt}x(t) = \sum_{i=1}^{r} \omega_i(z(t))A_i x(t) = \sum_{i=1}^{r} h_i(z(t))A_i x(t). \tag{3}
$$

Every function described by $A_i x(t)$ is called a sub-system, where

$$
\omega_i(z(t)) = \prod_{j=1}^{g} M_{i,j}(z_j(t)), \quad h_i(z(t)) = \frac{\omega_i(z(t))}{\sum_{i=1}^{r} \omega_i(z(t))}
$$

and $x(t)=[z_1(t), z_2(t), \cdots, z_g(t)], M_{i,j}(z_j(t))$ is the grade of the membership function of premise variable $z_j(t)$ in set $M_{i,j}$.

Suppose to all $t$, $\omega_i(z(t)) \geq 0$, $i=1,2,\cdots,r$, $\sum_{i=1}^{r} \omega_i(z(t)) > 0$, then $h_i(z(t)) \geq 0$, $i=1,2,\cdots,r$, and $\sum_{i=1}^{r} h_i(z(t)) = 1$ to all $t$.

### 2.2 Stability Condition of T-S Fuzzy System

Takagi and Sugeno have analyzed the stability via Lyapunov’s direct method, proved the stability condition of a fuzzy system, and gave a controller design method[5].

Firstly, the system model is described by a T-S fuzzy model, then a designed controller seeks a common symmetric matrix $P$ to satisfy the Lyapunov function; if there exists such matrix $P$, the fuzzy system is asymptotically stable.

**Theorem 1**[2]. If there exists a common positive definite matrix $P$ to all the fuzzy subsystems that satisfies

$$
A_i^T P + P A_i < 0, \quad i=1,2,\cdots,r
$$

then the fuzzy system (3) is asymptotically stable in the large to all the subsystems.

For simplicity, we rewrite $\hat{A}_{i,j} = A_i - B_i F_j$, by Lyapunov’s direct method, the sufficient condition of the local state feedback that ensures the asymptotical stability of the closed-loop fuzzy system is

$$
\hat{A}_{i,j}^T P + P \hat{A}_{i,j} < 0. \tag{5}
$$

The mainly relaxed stability result of continuous fuzzy system was given by Tanaka[7].

**Theorem 2**[3]. Assume that the number of rules that fire for all $t$ is less than or equal to $s$, where $1 \leq s \leq r$. The equilibrium of the continuous fuzzy control system (2) is asymptotically stable in the large if there exists a common positive definite matrix $P$ and a common positive semi-definite matrix $Q$ such that

$$
\left( \frac{\hat{A}_{i,j}^T P + P \hat{A}_{i,j} + (s-1)Q}{2} \right)^T P + P \left( \frac{\hat{A}_{i,j}^T P + P \hat{A}_{i,j} + (s-1)Q}{2} \right) - Q \leq 0, \quad i < j \tag{6}
$$

for all $i$ and $j$ excepting the pairs $(i,j)$.

The stability condition of Theorem 1 relies on each subsystem $A_i, B_i$ and $F_i$, $i=1,2,\cdots,r$, but the grades of the membership function is not taken into account.

In the work below, we will propose an extension of [8], with a different strategy from [7] and derive a different stability result. Due to

$$
\sum_{i=1}^{r} h_i(z(t)) \geq 0,
$$

then

$$
\sum_{i=1}^{r} h_i(z(t)) = 1, \quad i=1,2,\cdots,r, \tag{9}
$$

when $m \geq r$, $m$ is an integer variable. Then

$$
\sum_{i=1}^{r} h_i(z(t)) - \frac{2}{m-1} \sum_{i=1}^{r} h_i(z(t)) h_j(z(t)) \geq 0. \tag{10a}
$$
Assume that the number of fuzzy rules that fire for all \( t \) is \( n_0, 1 \leq n_0 \leq r \), when \( m \in \mathbb{R}^r \), (10b) is also equivalent. Because
\[
\left\{ \sum_{i=1}^{r} h_i(z(t)) \right\}^2 = \sum_{i=1}^{r} h_i^2(z(t)) + 2 \sum_{i=1}^{r} h_i(z(t)) h_j(z(t)) = 1. \quad (11)
\]
Substitute (11) to (10) and
\[
\frac{2r}{r-1} \sum_{i=1}^{r} h_i(z(t)) h_j(z(t)) \leq 1
\]
so
\[
\sum_{i=1}^{r} h_i(z(t)) h_j(z(t)) \leq \frac{r-1}{r}. \quad (12)
\]
Rewrite (2), we get
\[
\frac{d}{dt} x(t) = \sum_{i=1}^{r} h_i(z(t)) \hat{A}_{i,j} x(t) + 2 \sum_{i=1}^{r} h_i(z(t)) h_j(z(t)) \left\{ \hat{A}_{i,j} + \frac{\hat{A}_{i,j}}{2} \right\} x(t) \quad (13)
\]
and then obtain the following theorem.

**Theorem 3.** Assume that the number of rules that fire for all \( t \) is \( n_0, 1 \leq n_0 \leq r \), and is less than or equal to \( m \) where \( 1 < m \leq r \). If there exists the common matrix \( P > 0 \) and \( Q \geq 0 \) such that
\[
\hat{A}_{i,j}^T P + P \hat{A}_{i,j} + (m-1)Q < 0 \quad (14)
\]
\[
\left( \frac{\hat{A}_{i,j} + \frac{\hat{A}_{i,j}}{2}}{m} \right)^T P + P \left( \frac{\hat{A}_{i,j} + \frac{\hat{A}_{i,j}}{2}}{m} \right) - Q \leq 0, \quad i < j \quad (15)
\]
the equilibrium of the continuous fuzzy control system (13) is asymptotically stable in the large.

**Proof.** Select the Lyapunov function as bellow:
\[
V(x(t)) = x^T(t)Px(t)
\]
we get
\[
\frac{d}{dt} V(x(t)) = \frac{d}{dt} x^T(t)P \dot{x}(t) + x^T(t)P \frac{d}{dt} x(t) = \sum_{i=1}^{r} h_i(z(t)) h_j(z(t)) x^T(t)(\hat{A}_{i,j} - B F_j)^T P + P(\hat{A}_{i,j} - B F_j) x(t)
\]
\[
= \sum_{i=1}^{r} h_i^2(z(t)) x^T(t) \left( \hat{A}_{i,j}^T P + P \hat{A}_{i,j} \right) x(t) + 2 \sum_{i=1}^{r} h_i(z(t)) h_j(z(t)) x^T(t) \times
\]
\[
\left( \frac{\hat{A}_{i,j} + \frac{\hat{A}_{i,j}}{2}}{m} \right)^T P + P \left( \frac{\hat{A}_{i,j} + \frac{\hat{A}_{i,j}}{2}}{m} \right) x(t).
\]
From the conditions and (8)-(12), we get the following results:

3. Robust Fuzzy Controller Design in Flight Control System

A flight control system is a typical nonlinear, uncertainty parameter modeled and time varying system. It is not easy to keep stable and robust when designing controller in flight. We select the simple airplane as same as that in [9] and design the fuzzy controller with the proposed method.

The known conditions are: the height of flight \( h=3 \) km, and the mach number \( M_a=0.5 \), 0.8, and 0.9. The purpose is to design a feedback controller that could stabilize an airplane in the range of \( M_a \in [1.4, 1.0] \). The design procedure of fuzzy control law is shown as below.

We definite three fuzzy subsets \( M_1=0.5 \), \( M_2=0.8 \), \( M_3=0.9 \), considering the height of flight \( h \) is a constant. Fig. 1 shows the grade of the fuzzy membership function of the mach number.

![Fig. 1. Grade of membership function about mach number.](image)
Fuzzy rules according to mach number are taken as the premise variables.

The $i$th fuzzy rule: if $M_i$ is $M_{i1}$, then
\[
\frac{d}{dt} x(t) = A_i x(t) + B_i u(t), \quad i = 1, 2, 3
\]
\[
y(t) = C_i x(t), \quad i = 1, 2, 3
\]
\[
u(t) = -F_i x(t), \quad i = 1, 2, 3.
\]
The global controller is
\[
u = - \left[ h_1(M_a) F_1 + h_2(M_a) F_2 + h_3(M_a) F_3 \right] x(t)
\]
where $u$ is the control input of tuning angle of elevator. And
\[
A_i = \begin{bmatrix} a_{i11} & a_{i12} & 0 & a_{i14} & a_{i15} \\ a_{i21} & a_{i22} & 1 & 0 & 0 \\ a_{i31} & a_{i32} & a_{i33} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & a_{i42} & 0 & a_{i44} & 0 \end{bmatrix}
\]
\[
C_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & c_{i62} & c_{i63} & 0 & 0 \end{bmatrix}
\]
\[
B_i = \begin{bmatrix} b_{i1} & b_{i2} & b_{i3} & 0 & 0 \end{bmatrix}^T
\]
\[
u = [\Delta]^T.
\]
The following results show the solution of Theorem 3 ($m=1.5$):
\[
P = \begin{bmatrix} 0.4737 & -0.6447 & 0.1512 & 1.1041 & 0.1509 \\ -0.6447 & 189.3608 & -48.6127 & -309.6894 & -31.9169 \\ 0.1512 & -48.6127 & 14.4828 & 81.3015 & 57.5025 \\ 0.1509 & -31.9169 & 8.4616 & 57.5025 & 7.8687 \\ 0.0000 & -0.0056 & 0.0015 & 0.0096 & 0.0010 \\ -0.0056 & 1.7251 & -0.4550 & -2.9288 & -0.3158 \\ 0.0015 & -0.4550 & 0.1202 & 0.7727 & 0.0834 \\ 0.0096 & -2.9288 & 0.7727 & 4.9807 & 0.5384 \\ 0.0010 & -0.3158 & 0.0834 & 0.5384 & 0.0585 \end{bmatrix}
\]
\[
Q = 1000 \times \begin{bmatrix} 0.0000 & -0.0056 & 0.0015 & 0.0096 & 0.0010 \\ -0.0056 & 1.7251 & -0.4550 & -2.9288 & -0.3158 \\ 0.0015 & -0.4550 & 0.1202 & 0.7727 & 0.0834 \\ 0.0096 & -2.9288 & 0.7727 & 4.9807 & 0.5384 \\ 0.0010 & -0.3158 & 0.0834 & 0.5384 & 0.0585 \end{bmatrix}
\]
\[
F_1 = [-0.0293 8.4853 -2.0864 -14.2802 -1.5875]^T
\]
\[
F_2 = [-0.0159 4.9589 -1.1354 -7.8240 -0.8476]^T
\]
\[
F_3 = [-0.0181 5.8186 -1.3416 -9.1799 -0.9940]^T.
\]

Fig. 2 to Fig. 4 show the simulation results under the initial condition $x=[0 2/57.3 0 0 0]^T$. We can see that the results from Theorem 3 are better in performance evaluation, such as the state value, adjustment time and overshoot, and also a better robustness to reject the parameter perturbation. A different damping constant could be obtained through tuning the value of this variable, so it helps to achieve better control effectiveness.

4. Conclusions

This paper uses a different strategy to discuss the relaxed stability condition via Lyapunov’s direct method. The derived result is different from the previous papers, which is flexible to gain a better control performance by adjusting a variable, and can be utilized to search a better control rule in large range varied models. The application of the new strategy in flight control system shows that the obtained stability condition is useful to design a stable controller of a nonlinear system and the controller also has good robustness to reject the parameter perturbation of the plane.
References


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